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Abstract

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ON A CLASSICAL MODEL OF AN INDEFINITE METRIC

(Presented by Academician N. N. Bogolyubov, 18 III 1958)

In a previous work by N. N. Bogolyubov and ourselves ([1]), a method was proposed for using an indefinite metric in problems of quantum field theory. The aim of the present note is to clarify the meaning of the proposed device by considering a certain analogy constructed within the framework of classical field theory.

Let us consider two classical fields, for example, a complex field $\psi(x)$ and a real field $\chi(x)$, with interaction Lagrangian:

$$\mathcal{L}_{\text{int}} = g \int \psi^*(x)\psi(x)\chi(x)\, dx. \tag{1}$$

We shall regard the field $\psi(x)$ as a genuine physical field, and the field $\chi(x)$ as fictitious (in the sense of ([1]))* and represent it in the form of the expansion

$$\chi(x) = \sum_{(n)} c_n \varphi_n(x). \tag{2}$$

It is evident that the analogue of "fields with indefinite norm" in the classical theory will be fields with negative energy or, what is the same thing, fields with the opposite sign in the Lagrangian of the free field. In accordance with this, we now write the complete Lagrangian as follows:

$$\mathcal{L} = \int dx\{\partial_\mu\psi^*(x)\partial^\mu\psi(x) - M^2\psi^*(x)\psi(x)\}+$$

$$+\frac{1}{2}\sum_{(n)}\varepsilon_n \int dx\{\partial_\mu\varphi_n(x)\partial^\mu\varphi_n(x) - m_n^2\varphi_n^2(x)\}+$$

$$+g\sum_{(n)} c_n \int \psi^*(x)\psi(x)\varphi_n(x)\, dx \tag{3}$$

(we have denoted by M the mass of the field $\psi(x)$, and by m_n the masses of the fictitious fields $\varphi_n(x)$), where $\varepsilon_n = \pm 1$; fields with $\varepsilon_n = -1$ would correspond in the quantum theory to an indefinite metric.

Variation of (3) gives

$$(\Box - M^2)\psi(x) = -g \sum_{(n)} c_n \varphi_n(x)\psi(x) = -J(x), \tag{4,1}$$

$$(\Box - m_n^2)\varphi_n(x) = -g c_n \varepsilon_n \psi^*(x)\psi(x) = -j_n(x). \tag{4,2}$$

Proceeding now as in the usual derivation of the Yang-Feldman formalism, we may rewrite (4,2) in integral form:

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* One could, of course, assume that the field $\chi(x)$ also contains a "physical" component $\varphi(x)$. Then we would arrive at a field-theory model with an ordinary triple interaction.

$$\varphi_n(x) = \varphi_n^{\text{in}}(x) - \int D_{m_n}^{\text{ret}}(x - x')\, j_n(x')\, dx' \tag{5,1}$$

or

$$\varphi_n(x) = \varphi_n^{\text{out}}(x) - \int D_{m_n}^{\text{adv}}(x - x')\, j_n(x')\, dx', \tag{5,2}$$

where, as usual, the incoming fields $\varphi_n^{\text{in}}(x)$ obey the free equations and coincide with $\varphi_n(x)$ as $t \to -\infty$, while the outgoing fields $\varphi_n^{\text{out}}(x)$ as $t \to +\infty$. Adding and subtracting these equations and introducing the symmetric Green functions $\overline{D}_n(x)$ and the Pauli–Jordan commutator functions $D_n(x)$, for fields with masses m_n, by means of the usual relations

$$D_{m_n}^{\text{ret}}(x) = \overline{D}_n(x) + {}^1\!/_2 D_n(x); \qquad D_{m_n}^{\text{adv}}(x) = \overline{D}_n(x) - {}^1\!/_2 D_n(x) \tag{6}$$

we obtain, instead of (5),

$$v_n(x) \equiv \frac{\varphi_n^{\text{out}}(x) + \varphi_n^{\text{in}}(x)}{2} = \frac{1}{2}\int D_n(x - x')\, j_n(x')\, dx'; \tag{7,1}$$

$$u_n(x) \equiv \frac{\varphi_n^{\text{out}}(x) + \varphi_n^{\text{in}}(x)}{2} = \int \overline{D}_n(x - x')\, j_n(x')\, dx' - \varphi_n(x). \tag{7,2}$$

In accordance with the program outlined in (1), we want the fictitious fields, even if they do carry energy, momentum, and other dynamical characteristics, in any case not to be able to exchange them during the whole time of the collision with the physical fields. In other words, we want the asymptotic values at $t = +\infty$ and at $t = -\infty$ of such dynamical characteristics of the fictitious fields $\varphi_n(x)$ to coincide with one another. But all such characteristics (we assume, of course, that as $t \to \pm\infty$ the interaction is switched off by means of the adiabatic hypothesis) will, at $t = \pm\infty$, be expressed as sums (integrals) of terms of the form $\varphi_n^{\text{out}}(x)\varphi_n^{\text{out}}(x')$ and, respectively, $\varphi_n^{\text{in}}(x)\varphi_n^{\text{in}}(x')$.

Therefore, in order to satisfy our requirement, it is sufficient to impose the condition

$$\varphi_n^{\text{out}}(x)\varphi_n^{\text{out}}(x') - \varphi_n^{\text{in}}(x)\varphi_n^{\text{in}}(x') = 0. \tag{8}$$

Substituting into (8), instead of $\varphi_n^{\text{in}}(x)$, $\varphi_n^{\text{out}}(x)$, the fields $u_n(x)$ and $v_n(x)$ introduced in (7), we arrive at the condition equivalent to (8),

$$u_n(x)v_n(x') + v_n(x)u_n(x') = 0. \tag{9}$$

Thus, in order to satisfy the requirement that energy, etc., not be transferred to unphysical fields, it is sufficient to require that*

$$u_n(x) = \frac{\varphi_n^{\text{out}}(x) + \varphi_n^{\text{in}}(x)}{2} = 0 \tag{9a}$$

or

$$v_n(x) = \frac{\varphi_n^{\text{out}}(x) - \varphi_n^{\text{in}}(x)}{2} = 0. \tag{9b}$$

Returning now to equations (7), we see that the conditions (9a) and (9b) are in principle of a completely different character. Indeed, condition (9b) requires the integral of the physical fields $\psi(x)$ standing on the right-hand side of (7,1) to vanish. Therefore it proves to be a condition imposed also on the physical part of the system, and it is clear that its fulfillment can be achieved only if the physical part of the system possesses certain special properties. On the contrary, condition (9a) imposes no physical—

* In fact, it would have been necessary to require not that (9) be fulfilled at all points, but only that a certain kind of integrals of sums of terms of this form vanish. Since these linear combinations would be rather varied, a detailed analysis of the question of the existence of such a possibility would be quite complicated.

part of the system. Owing to the fact that in the right-hand side of (7.2), besides the integral over the physical fields $\psi(x)$, there also stands a nonphysical field $\varphi_n(x)$, for any $\psi(x)$ we can always satisfy (9a) by choosing $\varphi_n(x)$—the equations

(7.2) will then simply determine the nonphysical fields $\varphi_n(x)$ through integrals over the physical fields. Thus, we can always impose the condition (9a) on the system without fear of any contradictions arising*.

Making use of this circumstance, we shall impose the condition (9a), in order to exclude altogether the nonphysical fields $\varphi_n(x)$ and henceforth work with only one physical field $\psi(x)$. We then arrive at the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = g^2 \int \overset{*}{\psi}(x)\psi(x)\, K(x-x')\, \psi(x')\overset{*}{\psi}(x')\, dx\, dx' \tag{10}$$

and the equations of motion

$$(\Box - M^2)\psi(x) = -g^2 \int dx'\, \overset{*}{\psi}(x')\psi(x)\, K(x-x')\psi(x), \tag{11}$$

where the kernel

$$K(x-x') = \sum_{(n)}{}' \varepsilon_n c_n^2 \overline{D}_n(x-x') \tag{12}$$

is expressed in the form of a sum (or an integral, if one introduces a continuous set of fictitious fields) of symmetric Green functions $\overline{D}_n(x-x')$ with different masses m_n. It is clear that, by suitably disposing of the coefficients c_n and the sign factors ε_n, we can make $K(x)$ either singular or regular to any desired degree. The latter is possible, of course, only by admitting an "indefinite metric." In the case of a continuous mass spectrum, instead of (12) we obtain

$$K(x-x') = \int_0^\infty d(m^2)\, \rho(m^2)\, D_m(x-x'), \tag{13}$$

where the spectral function $\rho(m^2)$, in view of the remarks made, need not be positive.

Thus we see that, by excluding the nonphysical fields $\varphi_n(x)$ from the initially local Lagrangian (1) by means of the condition (9a), we arrive at a theory of a typically nonlocal form. This result is quite natural—it is possible to establish a direct analogy between it and the attempts of a number of authors** to exclude, by requiring the use only of half-sums of retarded and advanced potentials, every notion of photons from electrodynamics, formulating it as a pure action-at-a-distance theory. The resulting kernel (12) or (13) is not an arbitrary function of $(x-x')^2$, since, by virtue of the properties of the functions $\overline{D}_m(x)$, it is in any case restricted by the requirement

$$K\big((x-x')^2\big) = 0 \quad \text{for } (x-x')^2 < 0.$$

Let us emphasize that the nonlocal character of equations (11) is essentially connected with the imposed nonlocal condition (9a). Indeed,

* Let us note here that if we tried to achieve not the absence of exchange of energy, etc. between physical states and the absence of nonphysical states of energy, etc., then for this it would be necessary to impose both conditions (9a) and (9b) simultaneously. In light of what has been said, it is clear that this would not lead to a contradiction only if the physical part of the system possessed special properties.

** See, for example, ([2]), where references are given to the extensive preceding literature, beginning with the works of Ritz and Einstein.

since the functions $\bar{D}$ are Green' s functions of the Klein–Gordon equation, it could be shown that, by differentiating (11) the required number of times, one could, at any rate for a finite number of fictitious fields, return to a differential equation. However, such an equation would be subject to nonlocal boundary conditions, and the theory would remain nonlocal.

Let us note that in passing from the classical example considered here to the quantum case, one essentially new point arises. As we have shown, a nonlocal theory can be obtained from a local one by imposing certain additional conditions on the field quantities. In quantum theory these conditions can in principle be imposed either on the field operators, or, as an additional Lorentz condition in electrodynamics, be regarded as conditions on the admissible amplitudes of the state. In the usual construction of nonlocal theories the first path has always been chosen. But by imposing additional conditions on the field operators, we always risk coming into conflict with the commutation relations. It is precisely in this circumstance that we see the main cause of the failures of the nonlocal theories considered; it is no accident, for example, that the difficulties of the Kristensen-Møller-Bloch variant ([3]) turned out to be connected precisely with the noncommutativity of the field operators.

The general idea of the method proposed in ([1]) consists precisely in choosing the second path, which does not lead ([1]) to difficulties of this kind. Therefore one may hope that the method proposed in ([1]) will make it possible, in particular, to construct a consistent theory with nonlocal interaction. In this connection we would like to note that recently there have also appeared some experimental indications of the necessity of introducing a nonlocal interaction. Thus, Lee and Yang ([4]) have recently found that the experimental value of the Michel parameter in μ-meson decay can be readily explained if a nonlocal kernel is introduced into the four-fermion interaction, necessarily including negative coefficients c_n corresponding to fields with an indefinite metric.

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Note: Figure translations are in progress. See original paper for figures.