ON THE CAUCHY PROBLEM FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS

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Abstract

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MATHEMATICS

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ON THE CAUCHY PROBLEM FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS

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In the present note we consider the question of convergence of the simplest difference scheme approximating the system of equations

$$\frac{\partial U_\alpha}{\partial t} = \sum_{\beta=1}^{N} F_{\alpha\beta}(t,x)\,\frac{\partial U_\beta}{\partial x} + B_\alpha(t,x) \qquad (\alpha = 1, \dots, N), \tag{1}$$

where the functions $F_{\alpha\beta}$, B_α are analytic in a certain closed domain G of the plane (t,x). The initial conditions

$$U_\alpha(t_0,x) = \varphi_\alpha(x) \qquad (\alpha = 1, \dots, N) \tag{2}$$

are prescribed on the segment $G_0 \subset G$ of the line $t = t_0$, and $\varphi_\alpha(x)$ are analytic on this segment. From the results obtained there follow two theorems*, the first of which concerns the application of the method of successive approximations to the solution of the posed problem, while the second establishes (in a certain sense) its well-posedness. These theorems thus receive elementary proofs.

1. Consider the homogeneous system (3), obtained from (1) when $B_\alpha \equiv 0$. We write (3) and (2) in matrix form:

$$U'_t = F(t,x)\cdot U'_x; \tag{3a}$$

$$U(t_0,x) = \varphi(x), \tag{2a}$$

where $U = (U_\alpha(t,x))$; $F(t,x) = (F_{\alpha\beta}(t,x))$; $\varphi(x) = (\varphi_\alpha(x))$.

Let $\widetilde{G}$ be a closed domain in the space of two complex variables $(\tilde{t},\tilde{x})$, in which all coefficients $F_{\alpha\beta}(\tilde{t},\tilde{x})$ are analytic; $\widetilde{G}_0$ is the intersection of $\widetilde{G}$ with the plane

$\tilde{t} = t_0$. We restrict the domain $\widetilde{G}$ so that all $\varphi_\alpha(\tilde{x})$ are analytic in $\widetilde{G}_0$. Choose strictly inside $\widetilde{G}$ such a closed domain $\widetilde{D}$, adjoining the plane $\tilde{t} = t_0$, that the segment of a straight line joining an arbitrary point $(\tilde{t}, \tilde{x}) \in \widetilde{D}$ with the point $(t_0, \tilde{x})$ lies entirely in the domain $\widetilde{D}$; let $\widetilde{D}_0$ be the intersection of $\widetilde{D}$ with the plane $\tilde{t} = t_0$, D the intersection of $\widetilde{D}$ with the plane (t, x), and D_0 the intersection of D with the line $t = t_0$. Choose a sufficiently small $\sigma > 0$, take an arbitrary point $(t, x) \in D$, and construct a mesh with nodes (t_i, x_j) $(i = 0, 1, \dots, n;\ t_n = t,\ x_0 = x,\ x_n = x + \sigma)$ and with constant steps $t_{i+1} - t_i = h,\ x_{j+1} - x_j = \sigma/n = \varepsilon$.

* These theoremes were proved in (1) by a method based on the use of semi-ordered spaces of L. V. Kantorovi

Consider the system of equations

$$S_{0j}^{(n)} = \varphi(x_j);$$

$$\frac{S_{i+1,j}^{(n)} - S_{ij}^{(n)}}{h} = F(t_i, x_j)\frac{S_{i,j+1}^{(n)} - S_{ij}^{(n)}}{\varepsilon} \tag{4}$$

$$(i = 0, 1, \dots, n-1; \quad j = 0, 1, \dots, n-i-1),$$

approximating equation (3a) and the initial condition (2a).

Theorem 1. *The sequence of exact solutions of the systems of finite-difference equations (4), as $n \to \infty$, converges uniformly in some domain Q, adjacent to the segment D_0, to the solution of problem (3a), (2a); the resulting solution is analytic in this domain and is expressed in the form of the series*

$$U(t, x) =$$

$$= \varphi(x) + \sum_{m=1}^{\infty} \iint \cdots \int_{t_0 \le t^{(1)} \le \dots \le t^{(m)} \le t} F(t^{(m)}, x)\frac{d}{dx} \cdots F(t^{(1)}, x)\frac{d\varphi(x)}{dx}\, dt^{(m)} \cdots dt^{(1)}, \tag{5}$$

uniformly convergent in the domain Q.

I give the main points of the proof. The solution of system (4) for $i = n,\ j = 0$ can be represented in the form

$$S^{(n)}(t, x) = S_{n0}^{(n)} = \varphi(x) + \sum_{m=1}^{\infty} \Phi_{nm}(t, x), \tag{6}$$

where

$$\Phi_{nm}(t,x) = \sum_{0\le q_1<\dots<q_m\le n-1} F\left(t_0 + \frac{t-t_0}{n}q_m, x\right)\frac{\Delta}{\varepsilon}\cdots$$

$$\cdots F\left(t_0 + \frac{t-t_0}{n}q_1, x\right)\frac{\Delta}{\varepsilon}\varphi(x)\left(\frac{t-t_0}{n}\right)^m \quad \text{for } m = 1,\dots,n;$$

$$\Phi_{nm}(t,x) \equiv 0 \quad \text{for } m = n+1, n+2, \dots$$

Here Δ is the difference operator with respect to x: $\Delta f_{ij} = f_{i,j+1} - f_{ij}$; its action extends to all quantities standing to its right; the q_i take integer values within the indicated limits.

Put

$$a = \max_{\alpha=1,\dots,N}\max_{\widetilde{G}_0}|\varphi_\alpha(\tilde{x})|; \qquad A = \max_{\alpha,\beta=1,\dots,N}\max_{\widetilde{G}}|F_{\alpha\beta}(\tilde{t},\tilde{x})|;$$

$$\|S^{(n)}(\tilde{t},\tilde{x})\| = \max_{\alpha=1,\dots,N}\max_{\widetilde{D}}|S^{(n)}_\alpha(\tilde{t},\tilde{x})|.$$

Using the formula of finite increments and estimating the derivatives with the aid of Cauchy inequalities for the coefficients of a power series, it is easy to prove that all the series (6) possess the common majorant

$$a\sum_{m=0}^{\infty}\left(\frac{2ANL}{R}\right)^m, \tag{7}$$

where $L = |\tilde{t} - t_0|$; $R > 0$ is a constant depending on the choice of the domain $\widetilde{D}$ inside $\widetilde{G}$. The majorant (7) converges in the domain $\widetilde{Q}$, cut out from $\widetilde{D}$ by the inequality

$$|\tilde{t} - t_0| \le L' < \frac{R}{2AN} \tag{8}$$

(the domain $\widetilde{Q}$ is adjacent to $\widetilde{D}_0$).

After this it is proved without difficulty that the sequence $S^{(n)}(\tilde{t},\tilde{x})$ converges uniformly in the closed domain $\widetilde{Q}$ to the expression standing on the right-hand side of (5); since all $S^{(n)}$ are analytic in the closed domain

$\widetilde{Q}$, then, by the Weierstrass theorem, whence follows the analyticity of their limit $S(\tilde{t},\tilde{x})$ and the possibility of termwise differentiation of (5), as a result of which it is established that $S(t,x)$ is, in the domain Q, a solution of problem (3a), (2a) (Q denotes the intersection of $\widetilde{Q}$ with the plane (t,x)).

In the case of equations with constant coefficients, expansion (5) becomes the usual power series in $(t-t_0)$.

Remark. As is known, in all cases when the Cauchy problem is posed incorrectly, attempts at its numerical solution by the method of nets do not lead to any results. From what has been proved it is evident that the resulting divergence is explained by the action of rounding errors (at least under the condition that the restrictions imposed here on the equations and the initial conditions, and for the difference scheme under consideration, are satisfied).

2. It is easy to see that the application, for the solution of the problem posed, of the method of successive approximations according to the scheme

$$U^{(0)}(t,x) = \varphi(x);$$

$$U^{(n+1)}(t,x) = \varphi(x) + \int_{t_0}^{t} F(\tau,x)\, U_x'^{(n)}(\tau,x)\, d\tau \qquad (n=0,1,...) \tag{9}$$

formally gives the expression standing on the right-hand side of (5).

From Theorem 1 it follows:

Theorem 2. *The sequence of iterations (9) converges uniformly in the domain Q to the solution of problem (3a), (2a), and the solution is analytic and is written in the form (5).*

3. From the form of the majorant (7) and the linearity of the equations follows Theorem 3.

Theorem 3. *For a fixed domain $\widetilde{D}_0$ lying strictly inside $\widetilde{G}_0$, changes of the initial conditions $\varphi_\alpha(\tilde{x})$, small in modulus and analytic in the domain $\widetilde{G}_0$, lead to changes of the solution that are small in modulus and analytic in the domain $\widetilde{Q}$.*

In other words, in the indicated sense the problem posed is well-posed.

4. Everything set forth above also extends to equations of the form (1) with $B_\alpha \neq 0$; in this case the solution is written in the form

$$U(t,x) = S(t,x) + P(t,x), \tag{10}$$

where $S(t,x)$ is the solution of the corresponding homogeneous system (3) with initial conditions (2), which is given by formula (5), and $P(t,x)$ is the solution of system (1) with initial conditions

$$U_\alpha(t_0, x) \equiv 0 \qquad (\alpha = 1, \dots, N), \tag{11}$$

expressible in the form

$$P(t,x) = \int_{t_0}^{t} B(t^{(0)}, x)\, dt^{(0)} + \tag{12}$$

$$+\sum_{m=1}^{\infty} \iint \cdots \int_{t_0 \le t^{(1)} \le \cdots \le t^{(m)} \le t} F(t^{(m)}, x) \frac{d}{dx} \cdots F(t^{(1)}, x) \frac{d}{dx} \int_{t_0}^{t^{(1)}} B(t^{(0)}, x)\, dt^{(0)}\, dt^{(m)} \cdots dt^{(1)},$$

where

$$B(t,x) = (B_\alpha(t,x));$$

series (12) converges uniformly in the domain Q under the condition that all B_α are analytic in the domain $\widetilde{G}$.

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CITED LITERATURE

1. A. I. Fet, *Uspekhi Mat. Nauk*, **11**, no. 2 (68) (1956).

Note: Figure translations are in progress. See original paper for figures.