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Abstract

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FUNCTIONS ANALYTIC WITH RESPECT TO A HYPERBOLIC OPERATOR WITH TWO INDEPENDENT VARIABLES

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Mathematics

§ 1. The subject of this work is the extension of certain results of the theory of operator-analytic functions, obtained by M. K. Fage (^{1}) for functions of one variable, to functions of two variables that are analytic with respect to a hyperbolic operator of second order.

Let the hyperbolic operator

$$H = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y} + r(x, y)I \qquad (1)$$

be given in the square $D = (|x| + |y| < 1)$, whose sides are characteristics; moreover $p(x, y)$ and $q(x, y)$ have continuous partial derivatives of first order in D, and $r(x, y)$ is continuous. For $|x_0| < a \leq 1$, denote by $D(x_0, a)$ the square $|x - x_0| + |y| < a - |x_0|$ lying in D; by $D(x_0, a, \varepsilon)$ denote the intersection of $D(x_0, a)$ with the strip $|x - x_0| < \varepsilon$.

Definition 1. A function $f(x, y)$, given in the domain D, will be called **infinitely H-differentiable** in D if, for every $k = 0, 1, 2, 3, ...$, the functions $H^k f(x, y)$ have continuous partial derivatives up to the second order inclusive.

Definition 2. An infinitely H-differentiable function $f(x, y)$ in the domain D will be called **(H, x)-analytic** in D if, for every closed domain $R \subset D$, there exists a constant $C > 0$ (depending on f, H, and R) such that in R the inequalities

$$\left|\frac{\partial^s}{\partial x^s} H^k f(x, y)\right| \leq C^{2k+s}(2k + s)! \qquad (2)$$

hold for all $s = 0, 1;\ k = 0, 1, 2, ...$, with the possible exception of $k = s = 0$ (in the exponent $2k+s$, the number two is the order of the operator H with respect to the variable x).

By an **operator inverse to the operator** H we shall mean an integral operator K_{x_0} that transforms a function $\varphi(x, y)$, given and continuous in the domain

$D(x_0, 1)$, into the solution of the equation $Hu = \varphi(x, y)$ under zero initial conditions on the segment $x = x_0$. By $C^{(m)}(a, b)$ we denote the class of functions continuously differentiable m times on the interval (a, b). For $s = 0, 1,\ |x_0| < 1$, introduce the operator $K^{(s)}_{x_0}$, which transforms functions $\varphi(y) \in C^{(2-s)}(|x_0| - 1,\ 1 - |x_0|)$ into solutions of the equation $Hu = 0$ under the following initial conditions:

$$\text{if } s = 0, \qquad u(x_0, y) = \varphi(y), \qquad u'_x(x_0, y) = 0;$$

$$\text{if } s = 1, \qquad u(x_0, y) = 0, \qquad u'_x(x_0, y) = \varphi(y).$$

To study the structure of an (H, x)-analytic function $f(x, y)$, given in the domain D, for each $x_0 \in (-1, +1)$ we construct a system of functions $\varphi_{s,k}(y)$

by the formulas

$$\varphi_{s,k}(y) = \frac{\partial^s}{\partial x^s} H^k f(x, y)\Big|_{x=x_0}, \qquad s = 0, 1, \quad k = 0, 1, 2, \ldots^* . \tag{3}$$

Let us note here that $\varphi_{s,k}(y) \in C^{(2-s)}(|x_0| - 1,\ 1 - |x_0|)$, and on every segment $[-a, a] \subset (|x_0| - 1,\ 1 - |x_0|)$ the inequalities

$$|\varphi_{s,k}(y)| \le C^{2k+s}(2k + s)! \tag{4}$$

hold.

Transferring the scheme for constructing the theory of L-Taylor series for functions of one variable (1), we establish the validity of the following theorems.

Theorem 1. *Let the function $f(x, y)$ be (H, x)-analytic in D. Then, for any $x_0 \in (-1, +1)$ and $a \in (0, 1)$, there exists an $\varepsilon > 0$ ($\varepsilon < 1 - |x_0|$) such that in $D(x_0, a, \varepsilon)$ the function $f(x, y)$ has an expansion into an "(H, x)-series" :*

$$f(x, y) = \sum_{k=0}^{\infty} K^k_{x_0} \sum_{s=0}^{1} K^{(s)}_{x_0} \varphi_{s,k}(y). \tag{5}$$

Theorem 2. *The functions $\varphi_{s,k}(y)$ of formula (5) are determined by the function $f(x, y)$ according to formulas (3), i.e. every (H, x)-series is an "(H, x)-Taylor series."*

Theorem 3. *An (H, x)-series admits termwise H-differentiation any number of times.*

Theorem 4. *If on the diagonal $\Delta(x = x_0)$ of the square $D(x_0, 1)$ there is given a system of functions $\varphi_{s,k}(y) \in C^{(2-s)}(|x_0| - 1,\ 1 - |x_0|)$, satisfying conditions*

(4) on every closed segment $[-a, a] \subset \Delta$, then the function $f(x, y)$, constructed by formula (5), is (H, x)-analytic in the domain $D(x_0, a, \varepsilon)$.

Here ε $(0 < \varepsilon < 1 - |x_0|)$ is determined in an appropriate way for each $a \in (0, 1)$.

Definition 3. A system of functions $\varphi_{s,k}(y) \in C^{(2-s)}(|x_0| - 1,\ 1 - |x_0|)$, satisfying conditions (4), will be called a **defining system** for the (H, x)-analytic function $f(x, y)$ constructed by formula (5).

§ 2. Let $x_0 \in (-1, +1)$; denote by A_{H,x_0} the set of functions each of which is defined and (H, x)-analytic in some (its own) domain $D(x_0, a, \varepsilon)$. The set of functions A_{H,x_0} is a linear vector space. We introduce a topology in A_{H,x_0} as follows. A sequence of functions $f_m(x, y) \in A_{H,x_0}$ $(m = 0, 1, 2, ...)$ will be called **regularly convergent** if the conditions are satisfied: a) there is a common domain $D(x_0, a, \varepsilon)$ in which all $f_m(x, y)$ are defined and satisfy inequalities of the form (2); b) each $\frac{\partial^s}{\partial x^s} H^k$-derivative $\frac{\partial^s}{\partial x^s} H^k f_m(x, y)$ tends in this domain uniformly to some limit.

Along with the operator H, consider another operator $\overline{H}$ of the form (1); for simplicity of exposition we assume that both operators are given in the domain D. We construct the topological spaces A_{H,x_0} and $A_{\overline{H},x_0}$. They consist respectively of sums

$$f(x, y) = \sum_{k=0}^{\infty} K_{x_0}^{k} \sum_{s=0}^{1} K_{x_0}^{(s)} \varphi_{s,k}(y), \qquad g(x, y) = \sum_{k=0}^{\infty} \overline{K}_{x_0}^{k} \sum_{s=0}^{1} \overline{K}_{x_0}^{(s)} \varphi_{s,k}(y),$$

where $\varphi_{s,k}(y)$ and $\overline{\varphi}_{s,k}(y)$ are functions of the defining systems, while $\overline{K}_{x_0}$ and $\overline{K}_{x_0}^{(s)}$ are operators constructed for $\overline{H}$ in the same way as K_{x_0} and $K_{x_0}^{(s)}$ for H.

We now define the transformation $T = T_{\overline{H},x_0;\, H,x_0}$ of the space $A_{\overline{H},x_0}$ onto A_{H,x_0} by the formula $Tg(x, y) = f(x, y)$, if $f(x, y)$ and $g(x, y)$ are constructed according to one and the same defining system of functions $\varphi_{s,k}(y)$. We obtain mutually— a one-to-one and continuous mapping of the linear topological space $A_{\bar{H},x_0}$ onto A_{H,x_0}. Since

$$Hf(x, y) = \sum_{k=1}^{\infty} K_{x_0}^{k-1} \sum_{s=0}^{1} K_{x_0}^{(s)} \varphi_{s,k}(y), \qquad \bar{H}g(x, y) = \sum_{k=1}^{\infty} \bar{K}_{x_0}^{k-1} \sum_{s=0}^{1} \bar{K}_{x_0}^{(s)} \varphi_{s,k}(y),$$

these functions correspond to one another:

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*The dependence of these functions on x_0 is not explicitly indicated.

$$Hf(x,y) = T\bar{H}g(x,y).$$

Substituting here $g(x,y) = T^{-1}f(x,y)$, we obtain $H = T\bar{H}T^{-1}$.

We have arrived at the theorem:

Theorem 5. *All hyperbolic operators of the second order are locally equivalent.*

§ 3. Comparing the results formulated above with the results of M. K. Fage [1] for functions analytic with respect to an ordinary linear differential operator

$$L = \frac{d^n}{dt^n} + p_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \dots + p_0(t)I, \qquad a_1 < t < b_1, \tag{6}$$

with continuous coefficients, we come to the conclusions:

a) the role of the sequence of coefficients $a_0, a_1, a_2, \dots$ of the L-series

$$\sum_{k=0}^{\infty} a_k f_k(t, t_0)$$

passes to the defining system of functions (3);

b) each term $a_k f_k(t, t_0)$ of the L-series is replaced by a term of the (H,x)-series (5), but from this latter term it is impossible to isolate functions of any "H-basis."

§ 4. Functions analytic with respect to operators can be used in solving the Cauchy problem (in t) for equations of the form

$$L^p u(x,y,t) = H^m u(x,y,t) \tag{7}$$

(where L is the operator (6) of order n; H is the operator (1); p, m are natural numbers) with initial values

$$\frac{\partial^s}{\partial t^s} L^q u(x,y,t_0) = \varphi_{s,q}(x,y) \tag{8}$$

$$(s = 0,1,2,\dots,n-1;\ q = 0,1,2,\dots,p-1;\ a_1 < t_0 < b_1),$$

which are (H,x)-analytic functions in the domain D.

Theorem 6. *If $np \geq 2m$, then for every domain $D(x_0, a) \subset D$ one can indicate an $\varepsilon > 0$ such that, for $t_0 \in (a_1, b_1)$, in the domain $(|x - x_0| + |y| < a - |x_0|,\ t_0 - \varepsilon < t < t_0 + \varepsilon)$: 1) the solution of problem (7)–(8) is given by the formula*

$$u(x,y,t)=\sum_{k=0}^{\infty}\sum_{s=0}^{n-1}\sum_{q=0}^{p-1} f_{s+nq+kpn}(t,t_0)\cdot H^{km}\varphi_{s,q}(x,y)$$

(here $f_{s+nq+kpn}(t,t_0)$ are functions of the L-basis [1]*); 2) this solution is an (H,x)-analytic function in x,y and an L-analytic function in t; 3) the solution is unique in the class of such functions.*

In an analogous way one can construct functions analytic in x with respect to the parabolic operator

$$\frac{\partial}{\partial x}-\frac{\partial^2}{\partial y^2}+p(x,y)I.$$

In conclusion, I express my sincere gratitude to the supervisor of the present work, M. K. Fage, for posing the problem and for valuable guidance in the course of its execution.

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CITED LITERATURE

1. M. K. Fage, DAN, **112**, No. 6 (1957).

Note: Figure translations are in progress. See original paper for figures.

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