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Abstract

Full Text

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ON A PROPERTY OF SETS EFFECTIVELY DISTINCT FROM ALL Φ-SETS

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In the paper ([4]) (p. 135), by analogy with the concept of effective uncountability belonging to P. S. Novikov ([1,2]), the concept of a set effectively distinct from all Φ-sets was introduced. There it was also proved that every set (in particular, a CA-set) effectively distinct from all A-sets contains a perfect compact nucleus. In the present note one of the further properties of sets effectively distinct from all Φ-sets is considered. We use here the definitions adopted in ([4]).

1°. Definition. We shall say that a δs-operation $\widetilde{\Phi}$ **embraces** a δs-operation Φ (or say: $\widetilde{\Phi}$ is **embracing** relative to Φ) if there exists a mapping $\tau(k)$ of the natural number sequence onto itself such that, whatever the space R, for every sequence $\{F_1, F_2, \dots, F_n\}$ of sets F_n of the space R one can choose such a sequence $\{F'_1, F'_2, \dots, F'_k, \dots\}$ of sets F'_k which satisfies the following requirements: a) for every k, $F'_k = F_{\tau(k)}$; b)

$$\widetilde{\Phi}\{F'_1, F'_2, \dots, F'_k, \dots\} = \Phi\{F_1, F_2, \dots, F_n, \dots\}.$$

The concept of an embracing operation obviously has the property of transitivity.

We note that the A-operation embraces the operations of lower and upper limits.

Theorem 1. *If the operation $\widetilde{\Phi}$ embraces the operation Φ, and T is a set of the metric space R effectively distinct from all $\widetilde{\Phi}$-sets of the space R, then T is effectively distinct from all Φ-sets of this space.*

2°. The proof of Theorem II § 4 of the paper ([4]), as well as Theorem 1 of the present note, make it possible to prove the following theorem.

Theorem 2. *Let R be a metric space and let Φ be a δs-operation embracing the operation of lower limit $\underline{\Phi}$ and the operation of upper limit $\overline{\Phi}$. If T $(T \subset R)$ is effectively distinct from all Φ-sets of the space R, then T and $R - T$ contain, respectively, sets E_1 and E_2 possessing the following properties: a) there exist such discontinua D_1 and D_2 that $E_1 \subset D_1$, $E_2 \subset D_2$; b) E_1 and E_2 are absolute G_δ' s; c) E_1 is not separable from $R - T$ by any absolute F_σ-set, and E_2 is not separable from T by any absolute F_σ-set.*

Proof. By virtue of Theorem 1, T is effectively distinct from all $\underline{\Phi}$-sets, and also effectively distinct from all $\overline{\Phi}$-sets.

We note that Theorem II § 4 ([4]) and its proof, given by us for the operation $\underline{\Phi}$, remain valid also for $\overline{\Phi}$, as well as for every

δs-operation Φ, possessing the following property: let a sequence of sets $\{M_1, M_2, \dots, M_n, \dots\}$ and a set M be given; if for almost all n, $M_n \subset M$, then $\Phi\{M_n\} \subset M$; while if for almost all n, $M_n \supset M$, then $\Phi\{M_n\} \supset M$.

Without reproducing all the notation and arguments of § 4 of paper ([4]), let us recall only that we used two sequences $\{F_n^0\}$ and $\{F_n^1\}$ of closed sets of the space R, satisfying the conditions: $0 \subset F_n^0 \subset Y_\Phi \subset F_n^1 \subset Z_\Phi$ for every n. At the end of the proof just mentioned we obtained a discontinuum D, which is the aggregate of all points of the form $\nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}; F_1^{t_2}, \dots, F_{q_2}^{t_2}; \dots; F_1^{t_k}, \dots, F_{q_k}^{t_k}; F_1^{t_{k+1}}, \dots, \dots, F_{q_{k+1}}^{t_{k+1}}, \dots\}$, where ν is a function ensuring the effective distinction of T from Φ-sets; $t_1, t_2, \dots, t_k, t_{k+1}, \dots$ independently take the two values 0 and 1, and $q_1, q_2, \dots, q_k, q_{k+1}, \dots$ are natural numbers chosen in a definite manner.

Let Φ be a lower-limit operation. Denote by D_1 the discontinuum D constructed for this case in ([4]). By virtue of the properties of the function ν, the set $R-T$ contains precisely those points $\nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}; \dots; F_1^{t_k}, \dots, F_{q_k}^{t_k}; \dots\}$ of the discontinuum D_1 for which almost all upper indices $t_1, t_2, \dots, t_k, \dots$ take the value 1. It is easy to verify, in view of this, that $D_1(R-T)$ is a countable everywhere dense subset of the discontinuum D_1. Consequently, the set $E_1 = D_1 \cdot T$ is an absolute G_δ, distinct from all absolute F_σ-sets. Whatever the set M such that $E_1 \subset M \subset T$, we have $M \cdot D_1 = E_1$. It follows from this that E_1 is an absolute G_δ not separable from $R-T$ by any absolute F_σ-set.

Assuming, further, that Φ is an upper-limit operation and denoting in this case the discontinuum D by D_2, we see, by virtue of the properties of the function ν, that the set T contains precisely those points $\nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}; \dots; F_1^{t_k}, \dots, F_{q_k}^{t_k}; \dots\}$ of the discontinuum D_2 for which almost all indices $t_1, t_2, \dots, t_k, \dots$ take the value 0. From this, analogously to the preceding, we conclude that the set $E_2 = (R-T) \cdot D_2$ is an absolute G_δ, not separable from T by any absolute F_σ-set. The theorem is proved.

Let us note the special case of this theorem when Φ is an A-operation and T is a CA-set. This special case of Theorem 2 may be regarded as a certain supplement to Gurevich' s theorem (([3]), p. 51).

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Note: Figure translations are in progress. See original paper for figures.