BOUNDARY PROPERTIES OF HARMONIC FUNCTIONS IN THREE-DIMENSIONAL SPACE

1958

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Abstract

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MATHEMATICS

N. I. MOZHEROVA

BOUNDARY PROPERTIES OF HARMONIC FUNCTIONS IN THREE-DIMENSIONAL SPACE

(Presented by Academician S. L. Sobolev on 31 V 1957)

In three-dimensional space x, y, z we consider a finite domain D, bounded by a sufficiently smooth surface S. For this domain the Dirichlet and Neumann problems are solved. We investigate the question: if the differential properties of the function f are known, then what differential properties are possessed by the functions u and v, which are solutions of the Dirichlet problem

$$\Delta u = 0, \qquad u\big|_S = f$$

or of the Neumann problem

$$\Delta v = 0, \qquad \frac{\partial v}{\partial n}\bigg|_S = f.$$

The first investigations in this direction belong to Lichtenstein ([8]) and A. M. Lyapunov. Lyapunov' s results were developed in the works of N. M. Günter and Kh. L. Smolitskii ([1]); the indicated authors proved the following theorems:

Theorem A. If the surface S is sufficiently smooth and f is bounded and has bounded continuous derivatives up to order l $(l \geq 0)$, and the derivatives of order l are properly continuous with exponent λ $(0 < \lambda \leq 1)$, then the solution of the Dirichlet problem is bounded and has bounded continuous derivatives up to order l, and the derivatives of order l are properly continuous with exponent λ', where $0 < \lambda' < \lambda \leq 1$.

Theorem B. If the surface S is sufficiently smooth and f is bounded and has bounded continuous derivatives up to order l $(l \geq 0)$, and the derivatives of order l are properly continuous with exponent λ $(0 < \lambda \leq 1)$, then the solution of the Neumann problem is bounded, has bounded continuous derivatives up to order $l+1$, and the $(l+1)$-st derivatives are properly continuous with exponent λ' $(0 < \lambda' < \lambda \leq 1)$.

We extend the results formulated to the case of the metric L_p $(1 \le p \le \infty)$ in terms of the classes $W_p^{(r)} H^{(\alpha)}(M; D)$, in particular somewhat strengthening the case $p = \infty$.

Definition 1. We shall say ([3]) that a measurable function $f(x_1, x_2, \dots, x_n)$, defined on the whole space R_n, belongs to the class $H^{(r)}_{px_1}(M)$, $r > 0$, $1 \le p \le \infty$, if it satisfies the following condition. Represent r in the form $r = \bar{r} + \alpha$, where $\bar{r}$ is an integer, $0 < \alpha \le 1$. The function f is integrable with the p-th power on R_n, together with its Sobolev-generalized ([4]) partial derivatives with respect to x_1 up to order $\bar{r}$ inclusive. In addition, the derivative $\partial^{\bar{r}} f / \partial x_1^{\bar{r}}$ satisfies the inequality

$$\left\| \frac{\partial^{\bar{r}} f(x_1 + h, \dots, x_n)}{\partial x_1^{\bar{r}}} - 2 \frac{\partial^{\bar{r}} f(x_1, \dots, x_n)}{\partial x_1^{\bar{r}}} + \frac{\partial^{\bar{r}} f(x_1 - h, \dots, x_n)}{\partial x_1^{\bar{r}}} \right\|_{L_p} \le M |h|^\alpha \tag{1}$$

or

$$\left\| \frac{\partial^{\bar{r}} f(x_2 + h, x_2, \dots, x_n)}{\partial x_1^{\bar{r}}} - \frac{\partial^{\bar{r}} f(x_1, x_2, \dots, x_n)}{\partial x_1^{\bar{r}}} \right\|_{L_p} \le M |h|^\alpha, \qquad 0 < \alpha < 1. \tag{2}$$

If the function belongs to $H^{(r)}_{px_i}$ for all $i = 1, \dots, n$, then we shall say that it belongs to $H^{(r)}_p(M)$.

Definition 2. Let D be a domain of the n-dimensional space R_n of points $(x_1, x_2, \dots, x_n)$, and let D' be a domain whose closure belongs to D. We shall call an n-dimensional vector h admissible, translating D' within D, if all the vectors th, where $0 \le t \le 1$, translate D' within D. Let r be a nonnegative integer, $0 < \alpha \le 1$, and M a positive constant. We shall say that a function f, defined on D, belongs to the class $W_p^{(r)} H_p^{(r)}(M; D)$ if it is integrable in the p-th power together with its generalized derivatives (mixed and unmixed) up to order r inclusive, and if for every partial derivative of order r, $f^{(r)}$ (mixed or unmixed), the inequalities

$$\|f^{(r)}(m + h) - f^{(r)}(m)\|_{L_p} =$$

$$= \left(\int_{(D')} |f^{(r)}(m + h) - f^{(r)}(m)|^p \, dm \right)^{1/p} \le M |h|^\alpha, \qquad 0 < \alpha < 1; \tag{3}$$

$$\|f^{(r)}(m + h) - 2 f^{(r)}(m) + f^{(r)}(m - h)\|_{L_p} =$$

$$= \left(\int_{(D')} |f^{(r)}(m+h) - 2f^{(r)}(m) + f^{(r)}(m-h)|^p \, dm \right)^{1/p} \leq M|h|, \qquad \alpha = 1, \tag{4}$$

hold for all $D' \in D$ and all admissible h. We shall agree to consider that, for $0 < \varepsilon < 1$,

$$W_p^{(r)} H_p^{(1+\varepsilon)}(D; M) = W_p^{(r+1)} H_p^{(\varepsilon)}(D; M).$$

The classes $W_p^{(r)} H_p^{(\alpha)}(M; S)$ of functions given on a surface S are defined analogously. We shall not define them for the sake of brevity, but we note that in this case, when $\alpha = 1$, as well as when $\alpha < 1$, the definition of functions of the class $W_p^{(r)} H^{(1)}(M; S)$ includes an inequality analogous to (3).

Theorem 1. Let

$$S \in W_\infty^{(r+2)} H^{(\alpha)}(M_1), \qquad f \in W_p^{(r)} H_p^{(\alpha-1/p,\ \alpha-1/p)}(M; S),$$

$M > 0,\ M_1 > 0,\ 0 < \alpha - 1/p \leq 1,\ r$ an integer, $1 \leq p \leq \infty$. Then the harmonic function solving the Dirichlet problem

$$\Delta u = 0, \qquad u|_S = f,$$

belongs to the class

$$W_p^{(\bar{r},\bar{r},\bar{r})} H_p^{(\alpha,\alpha,\alpha)}(\bar{M}; D),$$

where

$$\bar{M} \leq c\{M + \|f\|_{L_p(S)}\},$$

and c depends on D and p, but does not depend on M and $\|f\|_{L_p(S)}$.

Theorem 2. Let

$$S \in W_\infty^{(r+2)} H^{(\alpha)}(M_1), \qquad f \in W_p^{(r)} H_p^{(\alpha-1/p,\ \alpha-1/p)}(M; S),$$

$M > 0,\ 0 < \alpha - 1/p \leq 1,\ r$ an integer, $1 \leq p \leq \infty$. Then the harmonic function solving the Neumann problem

$$\Delta v = 0, \qquad \left.\frac{\partial v}{\partial n}\right|_S = f,$$

belongs to the class

$$W_p^{(\bar{r}+1,\bar{r}+1,\bar{r}+1)} H_p^{(\alpha,\alpha,\alpha)}(\bar{M}; D),$$

where

$$\bar{M} \leq c\{M + \|f\|_{L_p(S)}\},$$

and c depends on D, p, α, but does not depend on M and $\|f\|_{L_p(S)}$.

Let us note the works of S. M. Nikol' skii ([5]), T. I. Amanov ([6]), and O. V. Besov ([7]), in which analogous results were obtained in the case of the circle and the half-space. In the case $p = \infty$ and when the boundary function f is properly continuous with exponent $\alpha = 1$, our result somewhat strengthens the corresponding result of Lyapunov, Günter, and Smolitskii.

Next, the solution of the Poisson equation is investigated, and the following theorem is established:

Theorem 3. If $S \in W_\infty^{(r+2)} H^{(\alpha)}$, $f \in W_p^{(\bar{r},\bar{r},\bar{r})} H^{(\alpha,\alpha,\alpha)}(M; D)$, then $u(x, y, z)$, the solution of the equation $\Delta u = f$ under the conditions $u|_S = 0$ or $\partial u/\partial n|_S = 0$, belongs to the class $W_p^{(\bar{r}+2,\bar{r}+2,\bar{r}+2)} H^{(\alpha,\alpha,\alpha)}(\overline{M}; D)$, where

$$\overline{M} < c\{\|f\|_{L_p(D)} + M\},$$

and c depends on D, p, α, but does not depend on M or $\|f\|_{L_p(D)}$.

For $p = \infty$ this theorem is proved in N. M. Günter' s book ([1]).

As was said above, the results of the present article are proved for domains in three-dimensional space, but it is not difficult to extend them to the n-dimensional case. In obtaining them we mainly followed the method set forth in the book ([1]), the method of potential theory.

Let us note that our results for $\alpha \neq 1$, $\alpha \neq 1/p$ (simultaneously) cannot be strengthened, as follows from the embedding theorem of S. M. Nikol' skii ([3]) and from the circumstance (proved by S. M. Nikol' skii) that, for $\alpha \neq 1$, the classes $W_p^{(r)} H_p^{(\alpha)}(S)$ and $H_p^{(r+\alpha)}(S)$ coincide, and any function $f \in W_p^{(r)} H_p^{(\alpha)}(g)$ can be extended beyond g in such a way that the extended function $\bar{f} \in H_p^{(r+\alpha)}$ ([2]).

All-Union Institute
of Scientific and Technical Information

Received
28 IX 1956

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Note: Figure translations are in progress. See original paper for figures.