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Abstract

Full Text

PHYSICS

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SOME STATIONARY RELATIVISTIC FLOWS

(Presented by Academician N. N. Bogolyubov on 10 IX 1957)

Let us consider adiabatic stationary flows of a medium possessing symmetry. The basic equations of these flows have the form ([1])

$$u\frac{d(wu)}{dr} + \frac{dw}{dr} = 0, \qquad u\frac{d\ln v}{dr} = \frac{du}{dr} + \frac{Nu}{r}, \qquad \sigma = \text{const}, \qquad w = w(v). \quad (1)$$

Here u is the 4-velocity; $w = \rho v + \rho v c^2$ is the heat content; v is the specific volume; σ is the entropy; $N = 0$; 1; 2, respectively, for one-dimensional, cylindrical, and spherical flows of the medium.

The solution of the equations has the form

$$\frac{w}{\theta} = \frac{w}{\sqrt{1 - a^2/c^2}} = w_0, \quad (2)$$

where w_0 is a certain initial heat content (the heat content at rest);

$$\frac{ar^N}{v\theta} = \frac{ar^N}{v\sqrt{1 - a^2/c^2}} = \text{const} = m = \frac{\bar{m}}{\bar{B}}; \quad (3)$$

$\bar{B} = 1$; 2π; 4π, respectively, for $N = 0$; 1; 2; a is the velocity of the medium.

Knowing the equation of state (the adiabatic equation) $w = w(v)$, from (2) and (3) one can determine the dependences $a = a(r)$ and $w = w(r)$. In this case the quantities w_0 (the total energy density) and $\bar{m}$ (the mass flux) must be specified.

Eliminating the quantity a from (2) and (3), we arrive at the equation

$$r^{2N} = \frac{m^2}{c^2}\frac{w^2 v^2}{w_0^2 - w^2}. \quad (4)$$

It is evident that r has a minimum $r_{\min}$, and the motion of the medium is defined in the interval $r_{\min} \le r < \infty$. For $dr^{2N}/dw = 0$ we have

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$$-\frac{d\ln w}{d\ln v} = 1 - \frac{w^2}{w_0^2} \tag{5}$$

(and in this case $d^2r^{2N}/dw^2 > 0$). Since

$$-\frac{d\ln w}{d\ln v} = \frac{\omega^2}{c^2}, \tag{6}$$

where ω is the speed of sound, then

$$\frac{\omega_k^2}{c^2} = 1 - \frac{w_k^2}{w_0^2}. \tag{7}$$

Here the subscript k denotes the values of ω and w in the "cross-section" $r = r_{\min}$. Substituting w_k/w_0 into equation (2), we find $1 - a_k^2/c^2 = 1 - \omega_k^2/c^2$, whence

$$a_k = \pm\omega_k, \tag{8}$$

which indicates the critical velocity of the flow of the medium at $r = r_{\min}$.

Let us now determine the properties of the flow as $r \to \infty$. It is obvious that in this case $v \to \infty$; for real equations of state, when $dp/dv < 0$, the quantity w decreases, while the quantity a increases as r increases from $r_{\min}$ to ∞.

It is meaningful to carry out the further calculation for a specific equation of the adiabat. Let this equation have the form

$$pv^k = A = \text{const}. \tag{9}$$

Then

$$\rho v = 1 - \frac{Av_a^{1-k}}{(k-1)c^2} + \frac{Av^{1-k}}{(k-1)c^2}, \tag{10}$$

where $v_a = \text{const}$;

$$w = \alpha c^2 + \frac{kAv^{1-k}}{k-1}; \qquad w_0 = \alpha c^2 + \frac{vAv_0^{1-k}}{k-1}; \tag{11}$$

$$\alpha = 1 - \frac{Av_a^{1-k}}{(k-1)c^2}; \qquad \frac{\omega^2}{c^2} = \frac{kAv^{1-k}}{w} = \frac{kAv^{1-k}}{\alpha c^2 + kAv^{1-k}/(k-1)}. \tag{12}$$

It follows from this that

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$$\frac{\omega^2}{c^2} = \frac{k-1}{w}\left(w - \alpha c^2\right). \tag{13}$$

At $r = r_{\min}$ we have, comparing (7) and (13):

$$\left(\frac{w}{w_0}\right)^2 - \frac{(k-1)\alpha c^2}{w} = 2 - k. \tag{14}$$

The dependence $r = r(w)$ takes the form

$$r^N = \frac{m}{c}\left(\frac{kA}{k-1}\right)^{1/(k-1)} \frac{w}{\sqrt{w_0^2 - w^2\left(w - \alpha c^2\right)^{1/(k-1)}}}. \tag{15}$$

As $r \to \infty$, $w \to \alpha c^2$, $\omega \to 0$, $v \to \infty$, $a = c\sqrt{1 - \alpha^2 c^4 / w_0^2}$, i.e. the quantity a has the maximum possible value.

In the case of an ultrarelativistic gas

$$\alpha = 1 - \frac{A v_a^{1-k}}{(k-1)c^2} = 0; \tag{16}$$

$$\rho v = \frac{A v^{1-k}}{(k-1)c^2} = \frac{pv}{(k-1)c^2}; \qquad w = k\rho v c^2 = \frac{kAv^{1-k}}{k-1}; \tag{17}$$

$$w_0 = k\rho_0 v_0 c^2 = \frac{kAv_0^{1-k}}{k-1};$$

$$\left(\frac{w_0}{w_k}\right)^2 = \left(\frac{v_k}{v_0}\right)^{2(k-1)} = \frac{1}{2-k}; \qquad \frac{\omega_k}{c} = \frac{a_k}{c} = \sqrt{k-1}; \tag{18}$$

$$r^N = \frac{m}{c}\left(\frac{kA}{k-1}\right)^{1/(k-1)} \frac{w^{(k-2)/(k-1)}}{\sqrt{w_0^2 - w^2}}. \tag{19}$$

$$\text{As } r \to \infty \qquad v \to \infty, \quad w \to 0, \quad a \to c. \tag{20}$$

In the case of an ordinary, non-ultrarelativistic gas, when

$$\frac{A v_a^{1-k}}{(k-1)c^2} = 0, \qquad \alpha = 1, \tag{21}$$

we have the relations

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$$\rho v = 1 + \frac{Av^{1-k}}{(k-1)c^2}; \qquad w = c^2 + \frac{kAv^{1-k}}{k-1}; \qquad w_0 = c^2 + \frac{kAv_0^{1-k}}{k-1}; \tag{22}$$

$$\left(\frac{w}{w_0}\right)^3 - (2-k)\frac{w}{w_0} = \frac{(k-1)c^2}{w_0}; \tag{23}$$

$$r^N = \frac{m}{c}\left(\frac{kA}{k-1}\right)^{1/(k-1)} \frac{w}{\sqrt{w_0^2 - w^2}\,(w-c^2)^{1/(k-1)}}. \tag{24}$$

As $r \to \infty$,

$$w \to c^2, \qquad v \to \infty, \qquad a = c\left(1 - \frac{c^4}{w_0^2}\right). \tag{25}$$

Let us show that, for a quasi-one-dimensional motion of a flow of a medium with variable cross-section s, a critical flow takes place in the smallest cross-section, i.e., that $|a_k| = \omega_k$.

In the region where $ds < 0$, we have $|a| < \omega$; in the region where $ds > 0$, we have $|a| > \omega$. The continuity equation in the present case will have the form:

$$\frac{as}{v\sqrt{1 - a^2/c^2}} = \bar{m}.$$

Since $\bar{m} = \text{const}$, it follows that

$$s\,d\frac{a}{v\sqrt{1-a^2/c^2}} + \frac{a}{v\sqrt{1-a^2/c^2}}\,ds = 0,$$

which gives

$$-\frac{dv}{v} + \frac{da}{a(1-a^2/c^2)} = -\frac{ds}{s}. \tag{26}$$

Since Bernoulli' s equation can be written in the form

$$-\frac{dw}{w} = \frac{a\,da}{c^2(1-a^2/c^2)} = \frac{\omega^2}{c^2}\frac{dv}{v}, \tag{27}$$

then, comparing (26) and (27), we shall have

$$\left(1 - \frac{a^2}{\omega^2}\right)\frac{da}{a(1-a^2/c^2)} = -\frac{ds}{s}. \tag{28}$$

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If $s = s_{\min}$, then $ds = 0$ and $|a_k| = \omega_k$. If $ds < 0$, then, indeed, $|a| < \omega$; if $ds > 0$, then $|a| > \omega$, which proves our assertion.

The amount of motion which a diverging flow can acquire (the total vector amount of motion is equal to zero, but if the directions of the individual streamlets are not taken into account, then the total scalar second amount of motion can be determined) is determined by the relation

$$F_n = \dot{J}_n = \bar{m} a_n, \tag{29}$$

where a_n in the general case can be calculated from Bernoulli' s equation

$$1 - \frac{a_n^2}{c^2} = \frac{w_n^2}{w_0^2}; \tag{30}$$

here

$$w_n = \alpha c^2 + \frac{kAv_n^{1-k}}{k-1} = \alpha c^2 + \frac{k}{k-1} A^{1/k} p_n^{(k-1)/k} \tag{31}$$

is the heat content of the external medium into which the outflow occurs.

For small values of p_n, the asymptotic dependences $v = v(r)$ and $p = p(r)$, on the basis of (15) and (31), for an ultrarelativistic gas have the form

$$r^N = \frac{mv_0}{c} \left(\frac{v_n}{v_0}\right)^{2-k} = \frac{mv_0}{c} \left(\frac{p_0}{p}\right)^{(2-k)/k}, \tag{32}$$

whence

$$\frac{v_n}{v_0} = \left(\frac{cr^N}{mv_0}\right)^{1/(2-k)}; \qquad \frac{p_n}{p_0} = \left(\frac{cr^N}{mv_0}\right)^{k/(k-2)}; \qquad \frac{a_n}{c} = 1 - \frac{1}{2} \left(\frac{cr^N}{mv_0}\right)^{2(k-1)/(k-2)}. \tag{33}$$

For an ordinary gas the dependence for the pressure has the form

$$\frac{p_p}{p_0} = \left(\frac{mv_0}{r^N}\right)^k \left(\frac{(k-1)v_0^{k-1}}{2kA}\right)^{k/2}. \tag{34}$$

Let us now give a numerical example for the spherical motion of an ultrarelativistic gas ($N = 2$). We take $k = {}^4/_3$; then we arrive at the equations

$$p = {}^1/_3 \rho c^2; \qquad pv^{4/3} = A = {}^1/_3 \rho v^{1/3} c^2;$$

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$$w = {}^4\!/_3 \rho v c^2 = 4Av^{-1/3}; \qquad w_0 = {}^4\!/_3 \rho_0 v_0 c^2 = 4Av_0^{-1/3}; \tag{35}$$

$$a_k = \omega_k = \frac{\sqrt{3}}{3}c; \qquad \left(\frac{w_k}{w_0}\right)^3 = \frac{v_k}{v_0} = \sqrt{\frac{27}{8}}; \qquad \frac{p_k}{p_0} = \frac{4}{9}; \qquad r^2_{\min} = \frac{3\sqrt{3}}{2}\frac{mv_0}{c}; \tag{36}$$

$$r^2 = \frac{mv_0}{a(1-a^2/c^2)} = \frac{mv_0}{c}\frac{v}{v_0}\left[\left(\frac{v}{v_0}\right)^{2/3} - 1\right]^{-1/2} = \frac{mv_0}{c}\frac{w_0^2}{w^2\sqrt{1-w^2/w_0^2}}; \tag{37}$$

$$\left(\frac{r}{r_{\min}}\right)^2 = \frac{2}{3\sqrt{3}}\frac{1}{\frac{a}{c}(1-a^2/c^2)} = \frac{2}{3\sqrt{3}}\frac{w_0^2}{w^2\sqrt{1-w^2/w_0^2}}. \tag{38}$$

Let us note that $r^2_{\min}$ depends only on the product mv_0, i.e. on a single quantity. A detailed study of the curve $r = r(a)$ shows that the motion is in fact defined in the interval $r_{\min} \leqslant r < \infty$. To each value of r there correspond two values of a (and w); the larger corresponds to the diverging flow, the smaller to the converging one (in this case one must replace a by $-a$ and m by $-m$). Formally there is also a solution defined in the interval $0 \leqslant r < \infty$, but then everywhere $-a > c$, which is impossible.

Analogous results also hold for the general case.

Let us now determine what the quantity w_0 is equal to for an ultrarelativistic gas. Obviously,

$$w_0 = kE_0 = \frac{k}{k-1}p_0 v_0 = kc^2 = \frac{kp_0}{(k-1)\rho_0}, \tag{39}$$

since $p_0 = (k-1)\rho_0 c^2$. In addition, since $pv = RT = \dfrac{R_0}{\mu_0}T$, we have

$$T_0 = \frac{\mu_0}{R_0}p_0 v_0 = \frac{\mu_0}{R_0}(k-1)c^2, \tag{40}$$

where μ_0 is the "molecular weight" of the gas particles; R is the gas constant; R_0 is the universal gas constant. In the case $k = {}^4\!/_3$ we have

$$w_0 = {}^4\!/_3 E_0 = 4p_0 v_0 = {}^4\!/_3 c^2, \tag{41}$$

whence

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$$p_0 v_0 = \frac{c^2}{3}; \qquad T_0 = \frac{\mu_0}{3R_0} c^2; \qquad \rho_0 v_0 = 1. \tag{42}$$

The statistics of an ultrarelativistic (photon) gas leads to the thermodynamic relations [2]

$$c_{v_0} T_0 = 4E_0 = 4c^2; \qquad T_0 \sigma_0 = 4 p_0 v_0; \qquad c_{v_0} = 3\sigma_0; \tag{43}$$

therefore

$$p_0 v_0 = {}^1/_{12} c_{v_0} T_0 = {}^1/_3 c^2 = {}^1/_3 \rho_0 v_0 c^2 \quad \text{and} \quad \rho v^{1/3} = v_0^{1/3}. \tag{44}$$

In conclusion we note that the limiting transition in the basic equations leads to the well-known relations of gas dynamics.

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CITED LITERATURE

1. K. P. Stanyukovich, *Unsteady Motions of a Continuous Medium*, ch. 15, 1955.

2. L. D. Landau, E. M. Lifshitz, *Statistical Physics*, § 60, 1951.

Note: Figure translations are in progress. See original paper for figures.

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