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Abstract

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STATISTICAL JUSTIFICATION OF THE EQUATIONS OF FILTRATION MOTION

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The principal feature of filtration motions is their statistical character. What is studied is not the motion of individual jets of liquid passing through separate pore channels, but the motion averaged over the cross section of the flow. In passing from the true motions taking place in individual pores to the averaged filtration motion, it is assumed that the liquid, as it were, occupies the entire space, including both the solid skeleton and the pore space. Scheidegger was the first to make a clear attempt to apply statistical methods to the study of filtration ([1,2]). However, the methods he proposed are very complicated, and the scheme considered does not fully correspond to the process being studied. Simpler results can be obtained by applying to the study of filtration motions the well-developed apparatus of the theory of turbulence.

A porous medium consists of particles of diverse sizes and shapes. Between the particles there are voids, through which the individual jets of liquid pass. The basic characteristic determining the distribution of velocities among the individual jets of liquid is the curve of distribution of the voids in the cross section of the porous medium according to their sizes. Each individual jet obeys the ordinary Navier–Stokes equations and flows with a velocity uniquely determined by these equations and by the boundary conditions. Differences in the characteristics of individual jets are determined only by the sizes and shapes of the pore voids.

Mean velocities do not obey the Navier–Stokes equations, and other equations must be derived for their determination. Consider a cross section of a porous medium which passes through a considerable number of particles forming the medium and intersects numerous jets of liquid. Each individual jet will have in the section its own dimensions and shape and its own velocity. These velocities are called **local** velocities.

The velocity averaged over the entire cross section S, including the area of the section of both the liquid jets and the skeleton, is called the **filtration** velocity. The total area of all jets is equal to

$$S_0 = \sum_{i=1}^{N} s_i = nS \tag{1}$$

where N is the number of jets in the section over which the averaging is performed, s_i is the cross-sectional area of an individual jet, and n is the porosity coefficient, $n < 1$.

The mean value of any function φ is determined by the formula

$$\overline{\varphi} = \frac{1}{S} \int_S \varphi \, dS. \tag{2}$$

For velocities that are nonzero only in the cross sections of individual streamlets, we obtain

$$\bar{\varphi} = n \frac{\sum \varphi_i s_i}{\sum s_i} \tag{3}$$

Representing the cross section of each streamlet as a circle of the same area and assuming that there is a continuous distribution of openings according to their sizes, in the limit we obtain

$$\bar{\varphi} = n \frac{\displaystyle\int_0^\infty \varphi(r) f(r) r^2 \, dr}{\displaystyle\int_0^\infty f(r) r^2 \, dr}, \tag{4}$$

where $f(r)$ is the distribution function of the openings according to their sizes; r is the radius of an opening.

The mean (filtration) velocity, equal to the ratio of the total fluid discharge to the cross-sectional area of the flow, is expressed by the formula

$$\bar{u} = n \frac{\sum u_i s_i}{\sum s_i} = n \frac{\displaystyle\int_0^\infty u(r) f(r) r^2 \, dr}{\displaystyle\int_0^\infty f(r) r^2 \, dr}. \tag{5}$$

We shall assume that the local velocities obey the Navier–Stokes equation, whose integral for a rectilinear streamlet may be written as

$$u = -\frac{\partial p}{\partial x} \frac{\alpha r^2}{\mu}, \tag{6}$$

where α is the shape coefficient of the channel cross section ($\alpha = 1/8$ for a circular channel), $\partial p/\partial x$ is the pressure gradient, and μ is the viscosity. The mean velocity according to (5) is

$$\bar{u} = -\frac{\partial p}{\partial x}\frac{\alpha}{\mu}\, n \frac{\int_0^\infty f(r) r^4\, dr}{\int_0^\infty f(r) r^2\, dr}. \tag{7}$$

Comparing this with the usual Darcy equation, we obtain an expression for the permeability coefficient k:

$$k = \alpha n \frac{\int_0^\infty f(r) r^4\, dr}{\int_0^\infty f(r) r^2\, dr}. \tag{8}$$

The ratio of the two integrals entering into this equality, having the dimension of the square of a linear size, may be taken as the square of some effective transverse size of the mean pore channel l_{e}.

This formula generalizes the analogous formula proposed by L. S. Leibenzon ([3]) and later used by M. D. Millionshchikov. In Leibenzon' s formula, instead of the porosity n, the pore area fraction equal to its mean value is introduced.

Passing to the formulation of the equations of the averaged motion, we shall start from the ordinary equations of motion in the form

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j}, \tag{9}$$

where u_i is the component of the local velocity along the axis x_i; τ_{ij} is the stress; ρ is the density; t is time.

To obtain equations relating not the local velocities u_i, but their averaged values, it is necessary to average system (9) over planes perpendicular to the direction of the axes. An analogous operation is performed in the theory of turbulence in order to obtain the averaged equations of motion (the Reynolds equations). The difference is that here each equation of the system has to be averaged over its own plane, whereas in the theory of turbulence all equations of the system are averaged in the same way.

It is necessary to make the assumption that the averaged quantities are independent of the choice of the direction of the averaging plane; this is satisfied if a random distribution of the particles forming the porous medium, and therefore also a random distribution of the local velocities, is assumed.

It is assumed that each local velocity u can be represented as the sum of the mean value $\overline{u}$ and a fluctuating deviation u'.

As a result of averaging we obtain:

$$\frac{\partial \overline{u_i}}{\partial t} + \sum_{j=1}^{3} \frac{\partial \overline{u_i}\,\overline{u_j}}{\partial x_j} + \sum_{j=1}^{3} \frac{\partial \overline{u_i' u_j'}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \sum_{j=1}^{3} \frac{\partial \overline{\tau_{ij}}}{\partial x_j}. \tag{10}$$

Equation (10) differs from (9) by the additional tensor

$$\sum_{j=1}^{3} \frac{\partial \overline{u_i' u_j'}}{\partial x_j},$$

which is the result of the nonuniformity of the distribution of true velocities over the cross section of the filtration flow. This tensor is analogous to the tensor of turbulent stresses and gives a certain additional resistance inherent in motion in porous media.

On the right-hand side of (10) stands the tensor of averaged stresses $\overline{\tau_{ij}}$. To pass from the tensor of averaged stresses to velocities, we apply the generalized Darcy hypothesis, asserting proportionality of this tensor and the components of the filtration velocity:

$$\sum_{j=1}^{3} \frac{\partial \overline{\tau_{ij}}}{\partial x_j} = -\frac{\mu}{k_i}\,\overline{u_i}, \tag{11}$$

where k_i is the permeability coefficient, having the dimension of area.

The origin of Darcy’ s law should be sought in the averaging of the motion of many fluid filaments having a length of order Δx, comparable with the dimensions of the particles forming the porous medium. Over this length the shape of a filament changes only slightly, and the pressure drop over its length satisfies formula (6), which is strictly valid for a rectilinear tube. For a tube of any other shape it is valid when the influence of inertia is neglected, since it takes into account the resistance determined only by viscous friction.

The obtained system (10) is not closed, since it contains the tensor of additional resistances of inertial origin. To close the system, one or another set of conditions must be introduced.

The tensor of additional resistances is the set of derivatives of the averaged values of products of the fluctuating–

of velocities $\overline{u_i', u_j'}$. As is known, the mean value of such a product is called the **correlation moment** and characterizes the degree of statistical connection between the varying quantities u_i' and u_j'. From the condition of randomness of the distribution of the characteristics of the porous medium it follows that the correlation between the projections of the velocity u on different axes must be vanishingly small, i.e. $\overline{u_i', u_j'} = 0$, if $i \neq j$. Then only the three diagonal terms of the tensor will be different from zero.

For an isotropic porous medium one may assume that the pulsation components do not depend on the choice of coordinate axes. Then, instead of (10), in the usual notation we obtain three equations of the form

$$\frac{D\overline{u}}{dt} = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \overline{u'^2}\right) - \frac{\mu}{k_x}\overline{u}. \tag{12}$$

Neglecting the inertial components and using the continuity equation, we obtain Laplace's equation for the sum of the static pressure and the velocity head of the pulsation velocities. Thus, for the case under consideration, the results of the usual theory of filtration are valid, but they must be corrected by the quantity $\overline{u'^2}$. For the one-dimensional problem we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\mu}{k}\overline{u} + \rho\frac{\partial \overline{u'^2}}{\partial x}.$$

Integrating, we find that the quantity $\overline{u'^2}$ varies along the filtration flow even in the case when its mean velocity does not change.

Indirect confirmation of this result for the one-dimensional problem was obtained by D. I. Leipunskaya and Ya. A. Pruslin ([4]).

To close the system one may introduce, on the basis of dimensional theory, the assumption

$$\frac{\partial \overline{u_{i'}^2}}{\partial x_i} = \frac{\overline{u'^2}}{l_i}, \tag{13}$$

where l_i is a new geometrical characteristic of the porous medium. The system (12)–(13) is closed, but contains two triples of geometrical parameters k_x, k_y, k_z and l_x, l_y, l_z, which must be determined from experiment.

Equations of the type (12) and (13) for one-dimensional flow were obtained earlier ([5,6]) and are used for investigating the filtration of liquids and gases under a nonlinear resistance regime. The validity of this equation has been proved by numerous experimental data.

It is important to note that in these equations the resistance of inertial origin enters alongside, and not instead of, the usual inertial components: $D\overline{u_i}/dt$.

The introduction of the geometrical parameters l_x, l_y, l_z, or, for one-dimensional motion, of one parameter l, determined empirically, is analogous to Prandtl' s basic hypothesis, which underlies all semi-empirical theories of turbulence. The quantity l is analogous to the Prandtl mixing length.

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REFERENCES

[1] A. E. Scheidegger, *J. Appl. Phys.*, **25**, No. 8 (1954).
[2] A. E. Scheidegger, *Canad. J. Phys.*, **34**, No. 7 (1956).
[3] L. S. Leibenzon, *Motion of Natural Liquids and Gases in a Porous Medium*, 1947.
[4] D. I. Leipunskaya, Ya. A. Pruslin, Abstracts of Reports at the All-Union Isotope Conference of the Academy of Sciences of the USSR, p. 166, 1957.
[5] E. M. Minskii, *DAN*, **78**, No. 3 (1951).
[6] E. M. Minskii, *DAN*, **103**, No. 3 (1955).

Note: Figure translations are in progress. See original paper for figures.