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Abstract

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ESTIMATES OF WEYL SUMS AND THE DISTRIBUTION OF PRIME NUMBERS

(Presented by Academician I. M. Vinogradov on 17 V 1958)

The present paper is a continuation of the papers ($^{1-3}$) and strengthens the results obtained in them.

Denote by $N_k^{(\nu)}(\lambda_1, \dots, \lambda_n)$ the number of solutions of the system of equations

$$x_1^\nu + \cdots + x_k^\nu = y_1^\nu + \cdots + y_k^\nu + \lambda_\nu, \qquad 1 \le x, y \le P \quad (\nu = 1, 2, \dots, n). \tag{1}$$

It is easy to show that for the trigonometric sum

$$S(\omega_1, \dots, \omega_n) = \sum_{x=1}^{P} e^{2\pi i(\omega_1 x + \cdots + \omega_{nx}{}^{n)}}$$

for any integer $k \ge 1$ the equality

$$|S(\omega_1, \dots, \omega_n)|^{2k} = \sum_{\lambda_1, \dots, \lambda_n} N_k^{(P)}(\lambda_1, \dots, \lambda_n) e^{2\pi i(\omega_1 \lambda_1 + \cdots + \omega_n \lambda_n)} \tag{2}$$

holds, where the summation is over all λ_ν $(\nu = 1, 2, \dots, n)$ satisfying the condition $|\lambda_\nu| < kP^\nu$. Further, from the definition of the quantities $N_k^{(P)}(\lambda_1, \dots, \lambda_n)$ we obtain without difficulty the relations

$$\sum_{\lambda_{s+1}, \dots, \lambda_n} N_k^{(P)}(\lambda_1, \dots, \lambda_n) = N_k^{(P)}(\lambda_1, \dots, \lambda_s). \tag{3}$$

Consider the Weyl sum $S = S(\alpha_1, \dots, \alpha_{n+1})$.

Fundamental lemma. Let $P_1 \le P$, $\beta_\nu = C_{\nu+1}^1 \alpha_{\nu+1} \lambda_1 + \cdots + C_{n+1}^{n+1-\nu} \times \alpha_{n+1} \lambda_{n+1-\nu}$ $(\nu = 1, 2, \dots, n)$, and

$$V_{kk_1} = \sum_{\lambda_1, \dots, \mu_n} N_k^{(P)}(\lambda_1, \dots, \lambda_n) N_{k_1}^{(P_1)}(\mu_1, \dots, \mu_n) e^{2\pi i(\beta_1 \mu_1 + \cdots + \beta_n \mu_n)},$$

where the summation is over all $|\lambda_\nu| < kP^\nu$, $|\mu_\nu| < k_1 P_1^\nu$. Then

$$\left|\frac{1}{2}S\right|^{4kk_1} \le P^{2k(2k_1-1)}P_1^{-2k}V_{kk_1} + (2P_1)^{4kk_1}.$$

The proof of the lemma is obtained with the aid of Hölder' s inequality and the relations (2), (3), from the estimate

$$|S| \le \frac{1}{P_1}\sum_{y=1}^{P_1}\left|\sum_{x=1}^{P} e^{2\pi i f(x+y)}\right| + 2P_1,$$

where $f(x) = \alpha_1 x + \cdots + \alpha_{n+1}x^{n+1}$.

Theorem 1. Let

$$\alpha_{n+1} = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a,q) = 1, \quad |\theta| < 1, \quad q = P^r;$$

let r belong to the interval

$$\sqrt{n}\ln n < r < n - \sqrt{n}\ln n.$$

Then there exist absolute constants C and $\gamma > 0$ such that

$$\left|\sum_{x=1}^{P} e^{2\pi i(\alpha_1 x + \ldots + \alpha_{n+1}x^{n+1})}\right| \ll CP^{1-\frac{\gamma}{n^2\ln n}}.$$

Proof. Write V_{kk_1} in the form

$$V_{kk_1} = \sum_{\mu_1,\ldots,\mu_n} N_{k_1}^{(P_1)}(\mu_1,\ldots,\mu_n)\sum_{\lambda_1,\ldots,\lambda_n} N_k^{(P)}(\lambda_1,\ldots,\lambda_n)e^{2\pi i(\beta_1\mu_1+\ldots+\beta_n\mu_n)}. \quad (4)$$

According to the definition of the quantities β_ν, the expression $\beta_1\mu_1 + \ldots + \beta_n\mu_n$ is a linear homogeneous function of the quantities $\lambda_1,\ldots,\lambda_n$, and, consequently, by virtue of (2) the inner sum in (4) is nonnegative. Hence, putting

$$N_{k,n}^{(P)} = \max_{\lambda_1,\ldots,\lambda_n} N_k^{(P)}(\lambda_1,\ldots,\lambda_n),$$

we obtain*

$$V_{kk_1} \ll N_{k_1,n}^{(P_1)}\sum_{\mu_1,\ldots,\mu_n}\sum_{\lambda_1,\ldots,\lambda_n} N_k^{(P)}(\lambda_1,\ldots,\lambda_n)e^{2\pi i(\beta_1\mu_1+\ldots+\beta_n\mu_n)} \ll$$

$$\ll N_{k_1,n}^{(P_1)}\sum_{\lambda_1,\ldots,\lambda_n} N_k^{(P)}(\lambda_1,\ldots,\lambda_n)\min\left(2k_1P_1, \frac{1}{(\beta_1)}\right)\cdots\min\left(2k_1P_1^n, \frac{1}{(\beta_n)}\right).$$

Choose

$$s = \min(r, n-r).$$

Applying, for $\nu \leq n-s$, the estimate

$$\min\left(2k_1 P_1^{\nu}, \frac{1}{(\beta_\nu)}\right) \ll 2k_1 P_1^{\nu},$$

by virtue of (3), we obtain from this

$$V_{kk_1} \ll (2k_1)^{n-s} P_1^{\frac{(n-s)(n-s+1)}{2}} N_{k_1,n}^{(P_1)} V_k',$$

where

$$V_k' = \sum_{\lambda_1,\dots,\lambda_s} N_k^{(P)}(\lambda_1,\dots,\lambda_s) \min\left(2k_1 P_1^{n}, \frac{1}{(\beta_n)}\right) \cdots \min\left(2k_1 P_1^{n-s+1}, \frac{1}{(\beta_{n-s+1})}\right).$$

Further, using the estimate

$$\sum_{x=Q+1}^{Q+T} \min\left(U, \frac{1}{(mx+\beta)}\right) \ll C_0\left(\frac{mT}{q}+1\right)(U + q\ln q),$$

where

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a,q)=1, \quad |\theta|<1$$

and C_0 is an absolute constant, we obtain

$$V_k' \ll N_{k,s}^{(P)} \sum_{\lambda_1,\dots,\lambda_s} \min\left(2k_1 P_1^{n}, \frac{1}{(\beta_n)}\right) \cdots \min\left(2k_1 P_1^{n-s+1}, \frac{1}{(\beta_{n-s+1})}\right) \ll$$

$$\ll (2k_1 C_0)^s 2^{sn} P_1^{\frac{n(n+1)}{2} - \frac{(n-s)(n-s+1)}{2}} N_{k,s}^{(P)};$$

$$V_{kk_1} \ll (2k_1 C_0)^n 2^{sn} P_1^{\frac{n(n+1)}{2}} N_{k_1,n}^{(P_1)} N_{k,s}^{(P)}. \tag{5}$$

* By (β) is denoted the distance from β to the nearest integer.

Choose $k_1 = [M_1 n^2 \ln n]$ and $k = [Ms^2]$, where M_1 and M are sufficiently large positive constants. Then, by I. M. Vinogradov’ s mean value theorem (4,[5]), we obtain

$$N_{k_1,n}^{(P_1)} \ll e^{c_1 n^3 \ln^3 n} P_1^{2k_1 - \frac{n(n+1)}{2} + \frac{1}{2}}, \qquad N_{k,s}^{(P)} \ll e^{c_1 s^3 \ln s} P^{2k - \frac{s^2}{4}},$$

where c_1 is an absolute constant. Hence, by virtue of (5), applying the main lemma and observing that $s > \sqrt{n}\ln n$, we obtain the assertion of the theorem without difficulty.

Let, as above,

$$\alpha_{n+1} = \frac{a}{q} + \frac{\theta}{q^2}, \qquad (a, q) = 1, \qquad |\theta| < 1, \qquad q = P^r.$$

The estimate of Theorem 1 can be strengthened if one restricts oneself to a somewhat less wide interval of variation of r.

Theorem 2. *Whatever fixed $\varepsilon > 0$ may be, there exist an absolute constant C and a constant $\gamma = \gamma(\varepsilon)$ such that, for $\varepsilon n < r < n - \varepsilon n$, the estimate*

$$\left| \sum_{x=1}^{P} e^{2\pi i(\alpha_1 x + \dots + \alpha_{n+1} x^{n+1})} \right| \ll C P^{1 - \frac{\gamma}{n^2}}. \tag{6}$$

The proof of Theorem 2 differs from the preceding proof only in that, instead of $k_1 = [M_1 n^2 \ln n]$, one should choose $k_1 = [M_1 n^2]$. Estimates of the form (6) were obtained by me earlier for the case of rational trigonometric sums (^{1}). In Theorem 2 these estimates are extended to the case of arbitrary Weyl sums.

Consider the polynomial

$$f(x) = \alpha_1 x + \dots + \alpha_{n+1} x^{n+1},$$

some of whose coefficients are rational:

$$\alpha_\nu = \frac{a_\nu}{q}, \qquad \nu = s+2,\ s+3, \dots, 3s, \qquad 1 \le s \le \frac{n+1}{3}.$$

Denote by Δ_s the determinant of order s,

$$\Delta_s = \left| C^i_{s+i+j} a_{s+i+j} \right|.$$

Theorem 3. *Let δ be an arbitrary fixed number from the interval $0 < \delta < 1/3$, $n\delta \le s \le \frac{n+1}{3}$, $s + 1 \le r \le 2s(1 - \delta)$, $q = P^r$, and $(\Delta_s, q) = 1$. Then there exist constants $C = C(\delta)$ and $\gamma = \gamma(\delta)$ such that*

$$\left| \sum_{x=1}^{P} e^{2\pi i(\alpha_1 x + \dots + \alpha_{n+1} x^{n+1})} \right| < C P^{1 - \frac{\gamma}{n^2}}.$$

The proof of the theorem is based on the main lemma and certain additional considerations concerning the quantities $N_k^{(P)}(\lambda_1, \dots, \lambda_n)$, which make it possible to estimate more sharply the sum V_{kk} from the main lemma. Theorem 3 has various applications. Thus, for example, with its aid the following assertions are obtained:

Theorem 4. *As $|t| \to \infty$, for the Riemann function $\zeta(s)$ the estimate*

$$\zeta(1+it) = O(\ln^{2/3}|t|). \tag{7}$$

Theorem 5. Let $\pi(x)$ be the number of primes not exceeding x. There exists a constant $a > 0$ such that the estimate

$$\pi(x) - \int_2^x \frac{du}{\ln u} = O\left(xe^{-a\ln^{3/5}x}\right). \tag{8}$$

holds.

Theorems 4 and 5 strengthen the assertions of the papers (^{2,6}). An analogous strengthening of the results is also obtained in the questions mentioned in (^{3}).

Proof correction note. After the present note had been submitted for publication, a paper by I. M. Vinogradov (^{7}) appeared, in which estimates (7) and (8) were also obtained. The estimates in (^{7}) are based on an inequality whose idea coincides with the idea of the inequalities first applied in the papers (^{1,2}) (see also (^{8,9})).

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Note: Figure translations are in progress. See original paper for figures.

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