MATHEMATICS

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Abstract

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ON λ-STABILITY OF SOLUTIONS OF OPERATOR EQUATIONS IN BANACH SPACE

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Let, for each λ from a simply connected domain Λ of the complex-variable plane, there be defined a linear completely continuous operator $A_{(\lambda)}$, mapping a Banach space E into $E_{(\lambda)} \subset E$, and suppose that $A_{(\lambda)}$ depends analytically on λ in the domain Λ in the sense of convergence in norm: in a neighborhood of each point $\lambda_0 \in \Lambda$

$$A_{(\lambda)} = \sum_{j=0}^{+\infty} (\lambda - \lambda_0)^j A_j, \tag{1}$$

where A_j are linear (bounded) operators. Suppose that $A_{(\lambda)}$ has at least one regular point in Λ. Then [1,2] $R_{(\lambda)} = (I - A_{(\lambda)})^{-1}$ is a meromorphic operator in Λ.

Denote by F_λ the subspace of the space E consisting of those elements f for which the equation

$$u - A_{(\lambda)} u = f \tag{2}$$

is solvable for fixed $\lambda \in \Lambda$. The solution of equation (2) at a regular point λ ($F_\lambda = E$) will be denoted by u_λ.

Definition. A value $\lambda_0 \in \Lambda$ will be called a **stability point** of the operator $A_{(\lambda)}$ if, for every $f \in F_{\lambda_0}$, there exists

$$\lim_{\lambda \to \lambda_0} u_\lambda = u_{(\lambda_0)} \tag{3}$$

in the sense of strong convergence*.

Here, obviously, $u_{(\lambda_0)}$ is a solution of equation (2) for $\lambda = \lambda_0$, and if the point λ_0 is regular, then $u_{(\lambda_0)} = u_{\lambda_0}$. We shall call the solution $u_{(\lambda_0)}$ λ-stable.

The stability points include, obviously, all regular points of the operator $A_{(\lambda)}$. But spectral points may also fail to be stable in the sense indicated above. For example, if $A_{(\lambda)} = \lambda A$, then no multiple pole of the resolvent is a stability point (this follows directly from Theorem 1 of the present note). The aim of the present note is to clarify criteria for stability of spectral points of the operator $A_{(\lambda)}$, depending on the parameter λ in general nonlinearly, and the properties of λ-stable solutions.

Let λ_0 be a spectral point of the operator $A_{(\lambda)}$, in a neighborhood of which $R_{(\lambda)}$ has the expansion

$$R_{(\lambda)} = \sum_{j=-m}^{+\infty} (\lambda - \lambda_0)^j R_j;$$

* It is not difficult to show that, if the limit (3) does not exist in the sense of strong convergence, then it also does not exist in the sense of weak convergence.

$\{u_j\}_{j=1}^n$ is a basis of the eigensubspace E_0 of the operator $A_{(\lambda_0)}$;
$\{V_j\}_{j=1}^n$ is a basis of the eigensubspace E_0^* of the adjoint operator $A^*_{(\lambda_0)}$ ($E_0^* \subset E^*$; E^* is the space of functionals adjoint to E).

From the expansions (1), (4) and the equalities

$$R_{(\lambda)}(I - A_{(\lambda)}) = (I - A_{(\lambda)})R_{(\lambda)} = I \tag{4}$$

it follows that

$$(I - A_0)R_k = \begin{cases} 0, & \text{for } k = -m; \\ I + \sum_{j=1}^{m} A_j R_{-j}, & \text{for } k = 0; \\ \sum_{j=1}^{m+k} A_j R_{k-j}, & \text{for } k > -m,\ k \neq 0; \end{cases} \tag{5}$$

$$R_k(I - A_0) = \begin{cases} 0, & \text{for } k = -m; \\ I + \sum_{j=1}^{m} R_{-j} A_j, & \text{for } k = 0; \\ \sum_{j=1}^{m+k} R_{k-j} A_j, & \text{for } k > -m,\ k \neq 0. \end{cases} \tag{6}$$

From the equalities (5), (6) it is clear that the operators $R_{-1}, R_{-2}, \dots, R_{-m}$ are of finite rank, and moreover

$$R_{-m} f = \sum_{i,j=1}^{n} \gamma_{ij} u_i V_j(f) \tag{7}$$

(γ_{ij} are scalars). With the aid of the same equalities (5), (6), it is easy to establish the following proposition.

Theorem 1. *Every simple pole λ_0 of the resolvent is a point of stability; moreover, the λ-stable solution $u_{(\lambda_0)}$ is uniquely singled out from the n-parameter family of solutions of the equation*

$$u - A_{(\lambda_0)}u = f \qquad (f \in F_{\lambda_0}) \tag{8}$$

by the conditions

$$V_j(A_1 u_{(\lambda_0)}) = 0 \qquad (j = 1, \dots, n). \tag{9}$$

If, however, λ_0 is a multiple pole of the resolvent and, at the same time, the functionals $\omega_j(f) = V_j(A_1 f)$ $(j = 1, 2, \dots, n)$ are linearly independent in E, then λ_0 does not belong to the points of stability.

In connection with this theorem, it is of interest to obtain a necessary and sufficient condition under which the eigenvalue λ_0 is a simple pole of the resolvent. Such a condition is contained in the following theorem.

Theorem 2. *In order that the pole λ_0 of the resolvent $R_{(\lambda)}$ be simple $(m = 1)$, it is necessary and sufficient that*

$$\det\{V_j(A_1 u_i)\} \neq 0, \tag{10}$$

or, equivalently, that the eigensubspaces E_0 and E_0^* have bases $\{u_i^0\}$, $\{V_i^0\}$, biorthogonal with respect to the operator A_1:

$$V_i^0(A_1 u_j^0) = \delta_{ij}. \tag{11}$$

Theorem 2 is a generalization of a known result of Goursat ([3]) on resolvent kernels corresponding to Fredholm equations.

Theorem 3. Let λ_0 be an eigenvalue of rank $n = 1$ (with total multiplicity of the pole $m \geqslant 1$); let u_1 be an eigenvector of the operator $A_{(\lambda_0)} = A_0$; and let V_1 be an eigenfunctional of the adjoint operator $A^*_{(\lambda_0)}$. Then a necessary and sufficient condition for λ_0 to belong to the stability points is that

$$V_1(A_s u_1) \neq 0, \tag{12}$$

where the index s is defined by the condition: $V_1(A_{sf})$ is the first functional, not identically equal to zero, in the sequence*

$$V_1(A_1 f), \quad V_1(A_2 f), \dots, \quad V_1(A_{kf}), \dots \tag{13}$$

In this case, the characteristic property distinguishing the λ-stable solution $u_{(\lambda_0)}$ from the family of solutions of equation (2) for $\lambda = \lambda_0$ and $f \in F_{(\lambda_0)}$ is the equality

$$V_1(A_s u_{(\lambda_0)}) = 0. \tag{14}$$

Proof. First of all, note that λ_0 is a stability point if and only if $R_{-k}f = 0$ $(k = 1, 2, \dots, m)$ for every $f \in F_{\lambda_0}$. But for this it is necessary and sufficient that, for all $f \in E$,

$$R_{-k}f = z_{m-k}V_1(f) \quad (k = 1, 2, \dots, m), \tag{15}$$

where z_{m-k} are certain fixed elements of the space E. The sufficiency of such representations of the operators R_{-k} is obvious. The necessity can easily be established by means of the equalities (5).

Suppose first that $s = 1$, i.e.

$$V_1(A_1 f) \neq 0.$$

If, in addition, $V_1(A_1 u_1) \neq 0$, then, by Theorems 1 and 2, λ_0 is a stability point and the λ-stable solution satisfies condition (14) for $s = 1$. Now let $V_1(A_1 u_1) = 0$ and, consequently (by Theorem 2), $m > 1$. Then (see (7)) $R_{-m}f = \gamma u_1 V_1(f)$ $(\gamma \neq 0)$ and

$$R_{-m+1}(I - A_0)f = \gamma u_1 V_1(A_1 f),$$

and, since $V_1(A_1 f) \neq 0$, R_{-m+1} is not representable in the form (15), and thus λ_0 is not a stability point.

Consider the case $s > 1$. Then $m > 1$ and

$$R_{-m+j}f = z_j V_1(f) \quad [1 \leqslant j < \mu = \min\{s, m\}]. \tag{16}$$

Indeed,

$$(I - A_0)R_{-m+1}f = \gamma A_1 u_1 \cdot V_1(f),$$

whence

$$R_{-m+1}f = u_1 V(f) + x_1 V_1(f) \quad (V \in E^*,\ x_1 \in E).$$

But

$$R_{-m+1}(I-A_0)f = \gamma u_1 V_1(A_1 f) = 0,$$

therefore $V = \alpha V_1$ (α is a scalar), and representation (16) for $j = 1$ holds. Similarly, if $\mu > 2$, then

$$(I-A_0)R_{-m+2}f = \gamma A_2 u_1 \cdot V_1(f) + A_1 z_1 \cdot V_1(f) = x_2 \cdot V_1(f),$$

$$R_{-m+2}(I-A_0)f = \gamma u_1 \cdot V_1(A_2 f) + z_1 \cdot V_1(A_1 f) = 0,$$

whence it follows that (16) is also valid for $j = 2$, and so on.

* It is easy to see that such an index s always exists: if, for all $k = 1, 2, \ldots$ and every $f \in E$, $V_1(A_{kf}) = 0$, then in some disk $|\lambda - \lambda_0| < \rho$ one would have $V_1(A_{(\lambda)}f) = V_1(A_{(\lambda_0)}f) = V_1(f)$, which is impossible, since the spectrum of the operator $A_{(\lambda)}$ is discrete.

We shall now show that $m \geq s$ and, consequently, $\mu = s$. If it were the case that $m < s$, then from (6) and (16) it would follow that

$$R_0^\circ(I-A_0)u_1 = u_1 + \sum_{j=1}^{m} R_{-j}A_j u_1 = u_1 + \sum_{j=1}^{m} z_{m-j}V_1(A_j u_1) = u_1,$$

which is impossible, since $(I-A_0)u_1 = 0$. Thus, $m \geq s$.

If $m = s$, then

$$0 = R_0(I-A_0)u_1 = u_1 + \sum_{j=1}^{m-1} z_{m-j}V_1(A_j u_1) + R_{-m}A_m u_1$$

$$= u_1\{1 + \gamma V_1(A_m u_1)\},$$

whence

$$\gamma V_1(A_m u_1) = \gamma V_1(A_s u_1) = -1,$$

i.e. $V_1(A_s u_1) \neq 0$.

If, however, $m > s$, then

$$0 = R_{-m+s}(I-A_0)u_1 = \sum_{j=0}^{s-1} R_{-m+j}A_{s-j}u_1$$

$$= \gamma u_1 V_1(A_s u_1) + \sum_{j=1}^{s-1} z_j V_1(A_{s-j} u_1) = \gamma u_1 V_1(A_s u_1),$$

and thus $V_1(A_s u_1) = 0$.

Thus, if $V_1(A_s u_1) \neq 0$, then $m = s$; if $V_1(A_s u_1) = 0$, then $m > s$. In the first case, as is seen from (16), λ_0 is a point of stability. In the second case ($m > s$),

$$R_{-m+s}(I - A_0)\hat{f} = \gamma u_1 V_1(A_s \hat{f}) \neq 0.$$

Consequently, R_{-m+s} cannot be represented in the form (15), and λ_0 is not a point of stability. Further, if $m = s$, then, obviously,

$$u_{(\lambda_0)} = R_0 \hat{f} \qquad (\hat{f} \in F_{\lambda_0}).$$

Moreover, from (5) it follows that

$$(I - A_0) R_s f = \sum_{j=1}^{2s} A_j R_{s-j} f = \sum_{j=1}^{s-1} A_j R_{s-j} f + A_s R_0^{\circ} f \qquad (f \in F_{\lambda_0}),$$

whence

$$V_1(A_s R_0 f) = V_1((I - A_0) R_s f) - \sum_{j=1}^{s-1} V_1(A_j R_{s-j} f) = 0 \qquad (f \in F_{\lambda_0}).$$

Thus, $V_1(A_s u_{(\lambda_0)}) = 0$, and, since $V_1(A_s u_1) \neq 0$, this condition uniquely singles out the λ-stable solution $u_{(\lambda_0)}$ from the one-parameter family of solutions, which completes the proof of the theorem.

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