MATHEMATICS

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1958

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Abstract

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GROUPS WITH FINITELY MANY CLASSES OF CONJUGATE ABELIAN SUBGROUPS

(Presented by Academician P. S. Aleksandrov, 1 VII 1957)

In Neumann' s paper ([1]) a class of groups was studied in which, for every subgroup, there exists only a finite number of its conjugate subgroups. In ([1]) it was proved that this class of groups coincides with the class of groups that are finite extensions of their centers.

The question arises: will the class of groups be broader in which not all classes of conjugate subgroups are finite, but only the classes of conjugate abelian subgroups?

This question was posed to the author by S. N. Chernikov, who also expressed the supposition that it has a negative answer. In the present paper the validity of this supposition is proved (Theorem 1).

Theorem 1. *A group in which, for every abelian subgroup, there exists only a finite number of its conjugate subgroups is a finite extension of its center.*

The converse of this theorem is trivial.

We shall preface the proof of the theorem with a number of lemmas.

Lemma 1. *If Theorem 1 is true for periodic groups, then it is true also for arbitrary groups.*

The proof of the lemma can easily be obtained by relying on the following proposition, due to S. N. Chernikov:

In a group $\mathfrak{G}$, all classes of conjugate elements are finite if and only if it is either itself locally normal, or is a central extension of an abelian group without torsion by means of a locally normal group.

Lemma 2. *If Theorem 1 is true for arbitrary p-groups (p an arbitrary prime number), then it is true also for arbitrary periodic groups.*

The proof of this lemma can be obtained without particular difficulty by relying on the following propositions, proved by the author in his diploma work.

1) *If in a periodic group $\mathfrak{G}$ the normalizer of every abelian subgroup has finite index, then the group $\mathfrak{G}$ contains only a finite number of non-invariant*

Sylow p-subgroups for the various p, and only a finite number of nonabelian ones among the invariant Sylow p-subgroups.

2) *If in an arbitrary group $\mathfrak{G}$ some Sylow p-subgroup $\mathfrak{P}$ determines a finite class of subgroups conjugate to it, then the intersection of all subgroups of this class has finite index in $\mathfrak{P}$.*

Lemma 3. *The p-group $\mathfrak{G}$, generated by the system of generators*

$$a_1,\ b_1,\ a_2,\ b_2, \dots,\ a_n,\ b_n, \dots$$

with defining relations

$$[a_i,\ a_j] = [b_i,\ b_j] = [a_i, b_j] = 1 \quad \text{for } i, j = 1, 2, \dots, \quad i \neq j;$$

$$[c_i,\ a_i] = [c_i,\ b_i] = 1, \quad o(c_i) = p \quad (c_i = [a_i, b_i]),$$

where $o(c_i)$ denotes the order of the element c_i, has an abelian subgroup with an infinite class of subgroups conjugate to it.

In the proof of this lemma one uses the possibility of decomposing the group generated by the elements a_i $(i = 1, 2, \dots)$ into a direct sum of cyclic summands.

Lemma 4. *An arbitrary locally normal p-group which is not a finite extension of its center has a subgroup with the properties described in Lemma 3.*

The process of singling out such a subgroup is analogous to Neumann' s process of singling out "H-subgroups" in those groups with finite classes of conjugate elements which are not finite extensions of their centers ([1]).

Lemmas 1-4 essentially exhaust the proof of Theorem 1. Indeed, Lemmas 1 and 2 reduce the proof to the case of p-groups. Further, if any one of the p-groups satisfying the condition of the theorem (such groups are, obviously, locally normal) is not a finite extension of its center, then, by Lemma 4, it has a subgroup with the properties described in Lemma 3. But such a subgroup contains, by Lemma 3, an infinite class of conjugate abelian subgroups, which contradicts the condition of Theorem 1.

Received
9 I 1957

REFERENCES

[1] B. H. Neumann, *Math. Zs.*, **63**, No. 1, 76 (1955).

Note: Figure translations are in progress. See original paper for figures.