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Abstract

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ASYMPTOTICS OF EIGENFUNCTIONS OF THE EQUATION $\Delta u + k^2 u = 0$ WITH BOUNDARY CONDITIONS ON EQUIDISTANT CURVES AND SCATTERING OF ELECTROMAGNETIC WAVES IN A WAVEGUIDE

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Consider the equation

$$\Delta\psi_k + k^2\psi_k = 0 \tag{1}$$

in a certain domain bounded: a) either by two closed equidistant curves (i.e., curves the distance between which along the normal is constant); b) or by two equidistant curves and two normals to them; c) or by two infinitely extending equidistant curves. Suppose that

$$\psi_k\big|_\Gamma = 0. \tag{2}$$

In the first two cases the spectrum will be discrete; in the last case, continuous.

Let us take as a new system of coordinates the arc length s of the inner equidistant curve (with positive curvature $\varkappa(s)$) and the length r of the normal to it. Let the distance between the equidistant curves be equal to a. Then

$$\psi_k(r,s) - \frac{\sin(\pi n r/a)}{\sqrt{(1-\varkappa(s)r)(1-\varkappa(s)a)}}\, z(s) = O\Big(\frac{a\gamma}{n}\Big)\,, \tag{3}$$

where

$$\gamma = \sqrt{k^2 - (\pi n/a)^2}, \tag{4}$$

and $z(s)$ satisfies the equation

$$z'' + \left\{\frac{\varkappa^2}{4} - \frac{\varkappa''}{4}\,\frac{a}{1-\varkappa a} - \frac{(\varkappa')^2}{3}\,\frac{a^2}{(1-\varkappa a)^2} + \gamma^2(1-\varkappa a)\right\} z = 0 \tag{5}$$

and, in case a), the periodicity condition, while in case b), zero boundary conditions.

The number n does not enter equation (1). It can, however, be specified by an additional condition (see below). Moreover, formula (3) may be regarded as a double asymptotic: first with respect to $a\gamma$—in which case n acquires a definite meaning ([1])—and then with respect to n. Formula (3) gives the asymptotics, as $k \to \infty$, of the eigenfunctions of equation (1), nonuniform with respect to γ, and therefore not for all eigenfunctions.

Let us outline the proof of formula (3). Denote

$$\Phi = \sin\frac{\pi n r}{a}\,\varphi(s)(1-\varkappa r)^{-1/2},$$

then

$$\Delta\Phi = \frac{1}{1-\varkappa r}\left\{\frac{\partial}{\partial r}(1-\varkappa r)\frac{\partial\Phi}{\partial r} + \frac{\partial}{\partial s}\left(\frac{1}{1-\varkappa r}\frac{\partial\Phi}{\partial s}\right)\right\} =$$

$$= -\left(\frac{n\pi}{a}\right)^2 \sin\frac{n\pi r}{a}\,\varphi(s)(1-\varkappa r)^{-1/2} + \sigma,$$

where

$$\sigma = \sin\frac{\pi n r}{a}(1-\varkappa r)^{-5/2}\left\{\frac{1}{4}\varkappa^2\varphi + \varphi'' + r(1-\varkappa r)^{-1}\left(\frac{1}{2}\varkappa''\varphi + 2\varkappa'\varphi'\right) + \frac{5}{4}r^2(\varkappa')^2(1-\varkappa r)^{-2}\varphi\right\}.$$

Hence

$$\Phi = \sum_k a_k\psi_k = \sum_k \frac{\int \psi_k\sigma\, d\Omega}{k^2-(\pi n/a)^2}\psi_k,$$

and consequently,

$$\Phi - \sum_{|k^2-(\pi n/a)^2|<\pi n/a} \frac{\int \psi_k\sigma\, d\Omega}{k^2-(\pi n/a)^2}\psi_k = O\left(\frac{a\varkappa}{n}\right).$$

In case c), summation over k must be replaced by integration.

To determine $\varphi(s)$, we form the secular equation. We shall seek the extremum of the functional $\int \psi_k\Delta\psi_k\, d\Omega$ under the condition $\int \psi_k^2\, d\Omega = 1$ in the class of functions

$$\Phi(s) = \sin\frac{\pi n r}{a}(1-\varkappa r)^{-1/2}\varphi(s).$$

The function $\varphi(s)$ satisfies the periodicity condition in case a) and the boundary conditions in case b),

$$\int \{\Phi\Delta\Phi + k^2\Phi^2\}\, d\Omega = \int ds \int_0^a (\Phi\sigma + k^2\Phi^2)(1 - \varkappa r)\, dr.$$

Replacing, approximately, $2\sin^2 \frac{\pi n r}{a}$ by unity* and integrating with respect to r, we obtain

$$\int \left\{ \frac{1}{2}\left[k^2 - \left(\frac{\pi n}{a}\right)^2\right] a\varphi^2(s) + \frac{1}{8}\frac{\varkappa^2(s)\varphi^2(s)a}{1-\varkappa a} + \frac{1}{2}\frac{\varphi''(s)a\varphi(s)}{1-\varkappa a} + \frac{1}{8}\frac{\varkappa''(s)\varphi^2(s)a^2}{(1-\varkappa a)^2} + \frac{1}{2}\frac{\varkappa'(s)\varphi'(s)a^2\varphi(s)}{(1-\varkappa a)^2} + \frac{5}{24}\frac{\varkappa'^2}{(}\right.$$

To determine $\varphi(s)$, we write the Euler equation

$$\varphi''(s) + \frac{\varkappa'(s)\varphi'(s)a}{1-\varkappa a} + \varphi(s)\left\{\frac{\varkappa^2(s)}{4} + \frac{\varkappa''(s)}{4}\frac{a}{1-\varkappa a} + \frac{5}{12}\varkappa'^2(s)\frac{a^2}{(1-\varkappa a)^2} + \gamma^2(1-\varkappa a)\right\} = 0.$$

Making the substitution $\varphi(s) = z(s)\sqrt{1-\varkappa a}$, we obtain (5).

Case c) can be applied to a plane curved waveguide, since the z-component of the electromagnetic wave satisfies equation (1) with condition (2).

Suppose, as usual ([1]), that the waveguide, as $s \to \pm\infty$, becomes a straight one. As is known, in this case the solution of equation (1) is subject to the conditions

$$u(s,r) \sim \sin\frac{\pi n_0}{a} r\, e^{i\gamma_0 s} + \sum_{n=0}^{\infty} A_n \sin\frac{\pi n}{a} r\, e^{-i\gamma_n s} \quad \text{as } s \to -\infty,$$

$$u(s,r) \sim \sum_{n=0}^{\infty} B_n \sin\frac{\pi n}{a} r\, e^{i\gamma_n s} \quad \text{as } s \to \infty. \tag{6}$$

Under these conditions the solution of the equations is unique ([1]). The problem consists in determining the reflection coefficients A_n and the transmission coefficients B_n.

Suppose that

$$a\gamma_0/n_0 \ll 1. \tag{6a}$$

* This is because $2\sin^2 \frac{\pi n r}{a} = 1 - \cos 2\frac{\pi n r}{a}$ converges weakly to unity as $n/a \to \infty$.

Then

$$u(s,r) - -\frac{\sin(\pi n_0 r/a)}{\sqrt{(1-\chi(s)r)(1-a\chi(s))}}\, z(s) = O\left(\frac{a\gamma_0}{n_0}\right),$$

where $z(s)$ satisfies equation (5) under the conditions

$$\begin{aligned} z(s) &\sim e^{i\gamma_0 s} + Ae^{-i\gamma_0 s} \quad \text{as } s \to -\infty, \\ z(s) &\sim Be^{i\gamma_0 s} \quad \text{as } s \to \infty. \end{aligned} \tag{7}$$

The physical meaning of problems (1), (2), (6) is as follows. The asymptotics as $k \to \infty$ corresponds to geometrical optics. $n/a\gamma \gg 1$ corresponds to a ray launched in the direction r. The case $n/a\gamma \sim 1$ corresponds to a ray launched at an angle to r; such a ray necessarily leaves the waveguide, and along the s-axis it always moves in one direction; this is valid with accuracy $O(\gamma^{-\infty})$[3], and therefore condition (6) excludes this case. By virtue of (6a), $k \gg 1$, consequently $\gamma^2 + n^2 \gg 1$, i.e. $n \gg 1$. Thus the asymptotics of the solution of equation (1) satisfying conditions (2), (6) will be expressed by formula (3). Condition (6) then passes into condition (7).

The determination of the coefficients A_n and B_n in problem (1), c), (2), (6) reduces to finding the coefficients A and B in problem (5), (7), i.e. to the ordinary quantum-mechanical problem of scattering by a potential barrier[4] in the one-dimensional case.

If the waveguide is straight or has the form of a ring, then the coefficients of equation (5) will be constant. If, however, the waveguide is glued together from straight and ring-shaped pieces, then the coefficients of (5) will be piecewise constant. In this case, to find A and B, one may use the formulas for determining the transmission and reflection coefficients through a piecewise-constant potential barrier (see, for example, (4), p. 95, problem 2).

If, in addition to the condition $n \gg 1$, we assume $\varkappa_0 \ll 1$, then one can apply the formula for the transmission coefficients over a small potential barrier.

Problem a) can be applied to the scattering of electromagnetic waves in a straight coaxial line with arbitrary cross section.

In the three-dimensional case, in a bent waveguide with arbitrary cross section, the asymptotics can be sought in the form $u\varphi(s)/\sqrt{J}$, where u is the asymptotic solution in the cross section, and J is the Jacobian of the transition to the corresponding coordinate system.

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CITED LITERATURE

[1] A. G. Sveshnikov, DAN, **110**, No. 2 (1956).
[2] A. Samarskii, A. Tikhonov, ZhTF, **27**, issue 11–12 (1947).
[3] M. Kline, Comm. Pure and Appl. Math., **4**, No. 2–3 (1951).
[4] L. Landau, E. Lifshitz, *Quantum Mechanics*, part 1, 1948.

Note: Figure translations are in progress. See original paper for figures.