ON THE ASYMPTOTICS OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR DIFFERENTIAL EQUATIONS

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

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ON THE ASYMPTOTICS OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR DIFFERENTIAL EQUATIONS

The method of constructing asymptotics with respect to a small parameter ε for solutions of boundary-value problems for linear differential equations ([1,2]) also carries over to certain classes of nonlinear differential equations. We shall illustrate this with the example of the ordinary equation

$$L_\varepsilon y \equiv \varepsilon y'' + \varphi(x, y)y' - \psi(x, y) = 0, \qquad y(0) = A, \quad y(1) = B. \tag{1}$$

The asymptotics of solutions of this problem in powers of the parameter A was studied by Vazov ([3]). Consider the limiting equation

$$L_0 w \equiv \varphi(x, w)w' - \psi(x, w) = 0. \tag{2}$$

Suppose that some domain D is covered by limiting curves $w = w(x)$ (i.e., by solutions of (2)). Then in D the quantities $w'(x) = \psi(x, y)/\varphi(x, y) = p(x, y)$ $(y = w(x))$ and $w''(x) = p'_x + p'_y p = q(x, y)$ are functions of (x, y).

Consider the case when in D

$$\varphi(x, y) \geq \gamma > 0, \tag{3}$$

which ensures, for solutions of (1), the appearance of a boundary layer in a neighborhood of $x = 0$. We shall call a curve $y = u(x)$ a curve *cutting from above* (*from below*) for solutions $\tilde{y}_\varepsilon(x)$ of equation (1) if, when $\tilde{y}_\varepsilon(x) \leq u(x)$ $(\tilde{y}_\varepsilon(x) \geq u(x))$, the line $y = \tilde{y}_\varepsilon(x)$ cannot have, inside the strip $0 < x < 1$, contact with $y = u(x)$. For this it is sufficient that

$$L_\varepsilon u \equiv \varepsilon u'' + \varphi(x, u)\,[u' - p(x, u)] < 0 \quad (> 0)$$

for $0 < x < 1$. (For example, a segment $y = \text{const}$ on which $\psi > 0$ will cut from above, or a limiting curve $y = w(x)$ on which $q < 0$.)

Suppose there exists in the strip $0 \le x \le 1$ a domain satisfying the following conditions (we shall call it an M-domain): 1) it is covered by a field of limiting curves $y = w(x)$ connecting points of the lines $x = 0$, $x = 1$; 2) it is bounded by the segments $[A_0, A_1]$ and $[B_0, B_1]$ of the lines $x = 0$ and $x = 1$ and by the curves r_1 and r_2, cutting from above and from below; 3) the segment $[B_0, B_1]$ of the line $x = 1$ contains a segment $[\overline{B}_0, \overline{B}_1]$ such that every limiting curve $y = w(x)$ issuing from a point $[1, B]$, $\overline{B}_0 \le B \le \overline{B}_1$, passes, for $0 \le x \le 1$, entirely inside this M-domain. It is easy to see that problem (1), for $A_0 < A < A_1$, $\overline{B}_0 < B < \overline{B}_1$, and sufficiently small $\varepsilon > 0$, has a solution $y = \tilde{y}_\varepsilon(x)$ passing inside the M-domain.

In what follows, unless otherwise stated, we shall assume that an M-domain exists and that the indicated inequalities are satisfied for the initial values A and B, and also that (3) is fulfilled. Note that every solution (1) lying in the M-domain is bounded: $|\tilde{y}_\varepsilon(x)| \le C$; hence it is easy to derive $|\tilde{y}'_\varepsilon(x)| \le C_1/\varepsilon$. In studying the solution $y = \tilde{y}_\varepsilon(x)$ it is convenient to use-

the function $z(x) = \tilde{y}'_\varepsilon(x) - p(x, \tilde{y}_\varepsilon(x))$. It satisfies the equation

$$\varepsilon z' = -\varphi_1(x, \tilde{y}_\varepsilon) z - \varepsilon q(x, \tilde{y}_\varepsilon), \qquad \varphi_1 = \varphi + \varepsilon p'_y. \tag{4}$$

Obviously, for sufficiently small ε, $\varphi_1 > \gamma_1 > 0$. Therefore, solving the Cauchy problem for (4) with initial conditions at $x = 0$, we find that $z(x)$ is the sum of an exponentially decreasing term of boundary-layer type and a term of order ε. Hence it follows (for sufficiently small ε) that the solution $\tilde{y}_\varepsilon(x)$ of problem (1), for $x > x_0$, where $x_0 = O(\varepsilon|\ln\varepsilon|)$, falls into the ε-neighborhood of the limiting curve $y = w(x)$ $(w(1) = B)$. For $0 < x < x_0$ the difference $v(x) = \tilde{y}_\varepsilon(x) - w(x)$ is a function of boundary-layer type, with $v(x_0) = O(\varepsilon)$, $v'(x_0) = O(\varepsilon)$. For $0 < x < x_0$, $v'(x) = O(1/\varepsilon)$, $\varepsilon v'' = O(1)$. Neglecting quantities of order $O(1)$, one may write for the principal part v_0 of this difference v the equation

$$\varepsilon v_0'' + \varphi(v_0 + a) v_0' = 0 \qquad (v_0(0) = A - a,\ a = w(0);\ \varphi(y) = \varphi(0, y)).$$

This equation is easily solved by quadratures and, as may be verified, for $\varphi \ge \gamma > 0$,

$$v_0(x) = O(1)\exp(-\gamma x/\varepsilon), \qquad v_0'(x) = O(1/\varepsilon)\exp(-\gamma x/\varepsilon).$$

Theorem. *If (3) holds in $\overline{M}$ and $\varphi(x, y)$ and $\psi(x, y)$ have the corresponding smoothness, then the following asymptotic representations hold for the solutions $\tilde{y}_\varepsilon(x)$ of problem (1) (where $A_0 < A < A_1$, $B_0 < B < B_1$), lying in M:*

$$\tilde{y}_\varepsilon(x) = w_0(x) + v_0(x) + \widetilde{R}_0(x), \qquad \widetilde{R}_0(x) = O(\varepsilon|\ln\varepsilon|), \tag{5}$$

$$\tilde{y}_\varepsilon(x) = \left[w_0(x) + \sum_{s=1}^{n} \varepsilon^s w_s(x)\right] + \left[v_0(x) + \sum_{s=1}^{n+1} \varepsilon^s v_s\right] + R_n(x),$$

$$R_n(x) = O(\varepsilon^{n+1}). \tag{6}$$

We present a scheme of the proof of formula (6). After separating out the principal terms $w_0(x) + v_0(x)$ of the asymptotics, we are able to linearize the equations determining the higher terms of this asymptotics. The construction of the asymptotics (6) is analogous to the process described in $(1, 2)$ for the linear case. We require that

$$L_\varepsilon \overline{w}_n = O(\varepsilon^{n+1}), \qquad \overline{w}_n = \sum_{0}^{n} \varepsilon^s w_s, \qquad w_0(1) = \tilde{y}(1), \quad w_s(1) = 0 \text{ for } s \geq 1. \tag{7}$$

Expanding at the point $(x, w_0(x))$ the functions $\varphi(x, \overline{w}_n)$ and $\psi(x, \overline{w}_n)$ in powers of ε and equating in (7) the terms with identical powers of ε, we obtain:

$$\varphi(x, w_0)w_0' - \psi(x, w_0) = 0, \qquad w_0(1) = B;$$

$$\varphi(x, w_0)w_k' + [\varphi_y'(x, w_0)w_0' - \psi_y'(x, w_0)]w_k = \Phi_k - w_{k-1}'', \qquad w_k(0) = 0; \tag{8}$$

Φ_k is a function of $w_0, w_1, \dots, w_{k-1}, w_0', \dots, w_{k-1}'$. Thus the w_k are successively determined by solving the linear equations (8); w_k, w_k', w_k'' are functions bounded on $[0, 1]$. To find the asymptotics of the boundary layer $\overline{v}_n = v_0 + \varepsilon v_1 + \dots + \varepsilon^{n+1} v_{n+1}$, we proceed from the equation

$$L_\varepsilon(\overline{v}_n + \overline{w}_n) - L_\varepsilon(\overline{w}_n) = O(\varepsilon^{n+1}), \qquad (\overline{v}_n + \overline{w}_n)\big|_{x=0} = A, \tag{9}$$

from which, in view of (7), it follows that $L_\varepsilon(\overline{v}_n + \overline{w}_n) = O(\varepsilon^{n+1})$. Introduce, as in $(1, 2)$, the variable $t = x/\varepsilon$; in this variable

$$\varepsilon L_\varepsilon u \equiv u''(t) + \varphi(\varepsilon t, u)u_t' - \varepsilon\psi(\varepsilon t, u).$$

Let us expand the function found, $\overline{w}_n = \sum_s^n \varepsilon^s w_s$, in a series in powers of $x = \varepsilon t$; recalling that $w(0) = a$, and grouping the terms according to powers of ε, we obtain:

$$\overline{w}_n(x) = \overline{w}_n(\varepsilon t) = a + \sum_{s}^{n} \varepsilon^s p_s(t) + O(\varepsilon^{n+1}), \tag{10}$$

where $p_s(t)$ are polynomials in t. Substituting this expression for $\overline{w}_n$ into (9) and expanding the coefficients $\varphi(\varepsilon t, \overline{w}_n)$ and $\psi(\varepsilon t, \overline{w}_n)$ in powers of ε, we successively obtain

$$v_0''(t) + \varphi(a + v_0)v_0'(t) = 0, \qquad v_0\big|_{t=0} = A - w_0(0) = A - a; \tag{11}$$

$$v_k''(t) + \varphi(a + v_0)v_k'(t) + \varphi_y'(a + v_0)v_0'v_k(t) = \Psi_k \quad (k = 0, 1, \dots, n + 1); \tag{12}$$

$$v_k(0) = -w_k(0) \quad \text{for } 1 \le k \le n; \qquad v_{n+1}(0) = 0,$$

where Ψ_k is a function of $v_0, v_1, \dots, v_{k-1}$, and also of $p_s(t)$, i.e. of already found functions. Here we seek v_k as functions of boundary-layer type ($v|_\infty = 0$), which replaces the second boundary condition. It is proved by induction that all functions v_k ($k = 0, 1, \dots, n + 1$) are functions of boundary-layer type.

To estimate $R_n(x)$ in formula (6), note that, in view of (7), (9), denoting $\tilde{y}_\varepsilon = \tilde{y}$, $\tilde{y}_1 = \tilde{y} - R_n (= \overline{v}_n + \overline{w}_n)$, we have:

$$L_\varepsilon \tilde{y} - L_\varepsilon \tilde{y}_1 = -L_\varepsilon \tilde{y}_1 = O(\varepsilon^{n+1}), \tag{13}$$

i.e.

$$\varepsilon R_n'' + [\varphi(x, \tilde{y})\tilde{y}' - \varphi(x, \tilde{y}_1)\tilde{y}_1'] - [\psi(x, \tilde{y}) - \psi(x, \tilde{y}_1)] = O(\varepsilon^{n+1}). \tag{14}$$

If $z_1 = \tilde{y}_1' - p(x, \tilde{y}_1)$, we obtain for z_1 an equation differing from (4) by the addition of $O(\varepsilon^{n+1})$ to the right-hand side. Denoting $\delta z = z - z_1$, we have:

$$R_n' = \tilde{y}' - \tilde{y}_1' = \overline{p}_y R_n + \delta z; \qquad \overline{p}_y = p_y(x, \tilde{y}_1 + \theta R_n), \quad 0 < \theta < 1. \tag{15}$$

Further, solving equation (4) and the corresponding equation for z_1, and noting that both $z(0)$ and $z_1(0)$ will be of order $1/\varepsilon$, we obtain

$$\delta z(x) = O(1/\varepsilon)\exp[-\gamma_1 x/\varepsilon] + O(\varepsilon)R_n(\theta x) + O(\varepsilon^{n+1}). \tag{16}$$

Considering (15) as a linear equation with respect to R_n, solving it under the condition $R_n(1) = 0$, and using (16), we obtain:

$$R_n(x) = O(1)\exp[-\gamma_1 x/\varepsilon] + O(\varepsilon)R_n(\xi) + O(\varepsilon^{n+1}), \tag{17}$$

where ξ is the point at which $|R_n(x)|$ attains its maximum. If $\xi \ge \varepsilon^{1-k}$, $0 < k < 1$, then from (17) it follows that

$$R_n(\xi) = O(1)\exp(-\gamma_1\varepsilon^{-k}) + O(\varepsilon^{n+1}). \tag{18}$$

Let $0 < \xi < \varepsilon^{1-k}$. Then, integrating equation (14) from ε^{1-k} to ξ, we obtain, since $R_n'(\xi) = 0$:

$$-\varepsilon R_n'(\varepsilon^{1-k}) + \int_{\tilde{y}(\varepsilon^{1-k})}^{\tilde{y}(\xi)} \varphi(y)\,dy - \int_{\tilde{y}1(\varepsilon^{1-k})}^{\tilde{y}_1(\xi)} \varphi(y)\,dy + \int_{\varepsilon^{1-k}}^{\xi} \Phi\,dx + O(\varepsilon^{n+1}) = 0, \tag{19}$$

where, as is easily verified, $\Phi = O(1)R_n + O(x)R_n'$, and

$$\int_{\varepsilon^{1-k}}^{\xi} \Phi\,dx = O(\varepsilon^{1-k})\,|R_n(\xi)|.$$

Further, $\varepsilon R_n'(\varepsilon^{1-k})$, by virtue of (15) and (16), is equal to $O(\varepsilon)|R_n(\xi)| + O(1)\exp[-\gamma_1\varepsilon^{-k}] + O(\varepsilon^{n+1})$. The difference of the integrals in (19) reduces to the integrals of $\varphi(y)$ over the interval $(\tilde{y}_1(\varepsilon^{1-k}), \tilde{y}(\varepsilon^{1-k}))$, of length $|R_n(\varepsilon^{1-k})| = O(1)\exp(-\gamma_1\varepsilon^{-k}) + O(\varepsilon)|R_n(\xi)| + O(\varepsilon^{n+1})$, and over the interval $(\tilde{y}_1(\xi), \tilde{y}(\xi))$, of length $|R_n(\xi)|$. We note that the integral over the second interval, which we denote by P, exceeds $\gamma|R_n(\xi)|$ in absolute value. The remaining terms in (19) give an expression of the form

$$O(1)\exp(-\gamma_1\varepsilon^{-k}) + O(\varepsilon^{1-k})|R_n(\xi)| + O(\varepsilon^{n+1}). \tag{20}$$

Hence, from (19), using the inequality $|P| \geq \gamma|R_n(\xi)|,\ \gamma > 0$, we obtain

$$|R_n(\xi)| = O(\varepsilon^{n+1}) + O(1)\exp(-\gamma_1\varepsilon^{-k}). \tag{21}$$

Since the second term in (21) is of higher order in comparison with the first, we obtain $|R_n(\xi)| = O(\varepsilon^{n+1})$. The theorem is proved. Analogously one proves:

If in the M-domain, under the fulfillment of the conditions of the theorem, there exist two solutions $\tilde{\tilde{y}}(x)$ and $\tilde{y}(x)$ of problem (1), then

$$\tilde{\tilde{y}}(x) - \tilde{y}(x) = O(\exp(-\gamma_1\varepsilon^{-k})),$$

where k is any fixed number between 0 and 1, i.e. uniqueness always holds up to a quantity exponentially small with respect to ε.

Sufficient conditions for uniqueness in the M-domain will be the simultaneous fulfillment of the inequalities:

$$\varphi > \gamma > 0, \qquad p_y > 0, \qquad (A - a)\varphi_y' \geq 0. \tag{22}$$

Remark. The constructions given above also carry over to equations of a more general form, for example $\varepsilon y'' + f(x, y, y') = 0$, under restrictions corresponding to those indicated above. It should be noted that in this case the boundary layer may have a weaker character of variation.

Remarks on quasilinear partial differential equations. As was shown in ([1], [2]), for the linear case the construction of the boundary layer reduces to the solution of an ordinary equation in the direction transverse to the boundary. Constructions of the same type also carry over to some classes of quasilinear elliptic partial differential equations. For example, for the equation $\varepsilon^2 \Delta u - \psi(\rho, \varphi, u) = 0$ under the conditions $u|_{\rho=0} = f(\varphi)$ ($\rho = 0$ is the equation of the boundary Γ), $\psi(\rho, \varphi, 0) = 0$, $\psi'_u > \gamma^2 > 0$, the solution of the limiting equation (for $\varepsilon = 0$) will be $w \equiv 0$, while for the boundary layer in the first approximation we obtain the ordinary equation

$$\varepsilon^2 A(\varphi) \frac{\partial^2 v}{\partial \rho^2} - \psi(0, \varphi, v) = 0, \qquad v|_{\rho=0} = f(\varphi),$$

which is analogous to (1). For the following approximations one obtains linear equations and an expansion of type (6) holds. In the same way one can obtain an asymptotic expansion of the form (6), for example, for quasilinear elliptic equations $L_\varepsilon u = h$ with a small parameter in the highest derivatives, if: 1) for arbitrarily small ε there exists and is unique a smooth solution $u_\varepsilon(x, y)$ of the boundary-value problem for $L_\varepsilon u = h$, which depends continuously (uniformly with respect to ε) on h (classes of such equations are easy to indicate, relying on the work of S. N. Bernstein ([4])); 2) the solution w of the limiting equation (for $\varepsilon = 0$) is sufficiently smooth; 3) the construction of the boundary layer reduces, for example, to an ordinary equation of the form (1).

Received
10 V 1958

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Note: Figure translations are in progress. See original paper for figures.