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Abstract

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ON THE THEORY OF THE IMPACT OF CYLINDERS

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The problem of the impact of circular cylinders identical in size and material was considered by A. N. Dinnikov ([1]). Below an approximate theory is set forth for the collision of arbitrary cylindrical bodies in the case of initial contact of the bodies of order $n > 1$.

We start from the simplest scheme: during the impact time t_2 the pressure region $2l$, the pressure $p(x,t)$ distributed over it, and the pressure force P remain unchanged, equal to their value under static compression for the case of initial contact of order n:

$$p(x_0,t_0) = \frac{P}{\pi l^2}\sqrt{l^2 - x_0^2}\left[\frac{2n}{2n-1} + \frac{2n(2n-2)}{(2n-1)(2n-3)}\frac{x_0^2}{l^2} + \cdots \right.$$
$$\left. \cdots + \frac{2n(2n-2)\cdots 2}{(2n-1)\cdots 1}\frac{x^{2n-2}}{l^{2n-2}}\right]; \tag{1}$$

$$p = \frac{(2n-1)!!}{(2n-2)!!} L_n l^{2n}; \qquad L_n = \frac{f_1^{(2n)}(0) + f_2^{(2n)}(0)}{(2n)!(\theta_1 + \theta_2)};$$

$f_j(x)$ determine the shape of the directing curves of the cylinders; $\theta_i = \dfrac{2}{\pi E_i}(1 - \nu_i^2)$; E_i is Young' s modulus; ν_i is Poisson' s constant.

Let us cut from the cylinders disks of thickness $h = 1$. Let m_1 and m_2 be their masses, and let $\alpha(t)$ be the approach of the axes. Then

$$M\frac{d^2\alpha}{dt^2} = -hP, \qquad M = \frac{m_1 m_2}{m_1 + m_2}, \qquad 0 \le t \le t_2. \tag{2}$$

Integrating with the condition $\alpha(0) = 0$, we find

$$\alpha(t) = v_0 t - \frac{hP}{2M}t^2, \qquad v_0 = \left(\frac{d\alpha}{dt}\right)_{t=0}. \tag{3}$$

If $t_1 = \frac{1}{2}t_2$ is the moment of completion of the first act of impact, then $(d\alpha/dt)_{t_1} = 0$, and we obtain:

$$t_1 = \frac{Mv_0}{hP}, \qquad \alpha(t_1) = \alpha_{\max} = \alpha_0 = \frac{1}{2}\frac{Mv_0^2}{hP}. \tag{4}$$

To find the unknowns t_1, P, l, let us establish one more relation between them by turning to the vibrations that arise during impact in the contacting sections of the disks. In the calculations these sections are replaced by half-planes. For the first half-plane, in the absence of initial data and body forces, we obtain for the displacement in the direction of the Oy axis ([2])

$$v_1(x_0, y_0, t_0) = \frac{1}{2\pi\rho_1}\left\{\frac{\partial}{\partial y_0}\left[\iint_{S_1} v_1^0 p(x,t)\,dx\,dt\right] - \frac{\partial}{\partial x_0}\left[\iint_{S_1} v_2^0 p(x,t)\,dx\,dt\right]\right\}, \tag{5}$$

where v_i^0 are analogous displacements determined by the fundamental solutions for the half-plane.

It is not difficult to justify the possibility of bringing the derivative sign under the integral sign and to write finally, for both half-planes,

$$v_j(x_0, 0, t_0) = \frac{1}{\pi\rho_j b_j^4}\iint_{S_j} \operatorname{Re}\frac{i\sqrt{1/a_j^2 - \theta^2}}{F_j(\theta)}\frac{p(x,t)}{x_0 - x}\,dx\,dt \qquad (j = 1, 2), \tag{6}$$

where $\theta = \dfrac{t_0 - t}{x - x_0}$; S_j are determined by the inequalities $r_j \leqslant a_j(t_0 - t)$, $0 \leqslant t \leqslant t_0$; a_j are the velocities of longitudinal waves; $F_j(\theta)$ is the Rayleigh function.

On the contact segment we easily obtain, as in the static case, the basic relation:

$$v_1(x_0, 0, t) + v_2(x_0, 0, t) = \alpha(t) - f_1(x_0) - f_2(x_0). \tag{7}$$

Substituting v_1 and v_2 from (6), we shall have:

$$\sum_{j=1}^{2}\frac{1}{\pi\rho_j b_j^4}\iint_{S_j} \operatorname{Re}\frac{i\sqrt{1/a_j^2 - \theta^2}}{F_j(\theta)}p(x,t)\frac{dx\,dt}{x_0 - x} = \alpha(t_0) - f_1(x_0) - f_2(x_0). \tag{8}$$

Put $x_0 = 0$, and let $a_{jt}0/l < 1$; then S_j will be the triangle shown in Fig. 1. Expand (1) in a series:

$$p(x_0, t_0) = \frac{P}{\pi l}\left[\frac{2n}{2n-1} + B_2(n)\frac{x_0^2}{l^2} + B_4(n)\frac{x_0^4}{l^4} + \cdots\right], \tag{9}$$

where $B_2, B_4, \ldots$ are known coefficients. Substituting in (8), we obtain ($f_1(0) = f_2(0) = 0$)

$$\frac{P}{\pi^2 l}\sum_{j=1}^{2}\frac{1}{\rho_j b_j^4}\int_0^{t_0}\int_{-a_j(t_0-t)}^{a_j(t_0-t)} \mathrm{Re}\,\frac{i\sqrt{1/a_j^2-\theta^2}}{F_j(\theta)}\times$$

$$\times\left[\frac{2n}{2n-1} + B_2(n)\frac{x^2}{l^2} + \cdots\right]\frac{dx}{-x} = \alpha(t_0). \tag{10}$$

Fig. 1

Putting $y = \dfrac{x}{t_0 - t}$, we shall have

$$\int_0^{a_j(t_0-t)} \mathrm{Re}\,\frac{i\sqrt{1/a_j^2-\theta^2}}{F_j(\theta)}\frac{dx}{-x} = \int_0^{a_j} \mathrm{Re}\,\frac{i\sqrt{1/a_j^2-1/y^2}}{F_j(1/y)}\frac{dy}{-y};$$

$$\int_0^{a_j(t_0-t)} \mathrm{Re}\,\frac{i\sqrt{1/a_j^2-\theta^2}}{F_j(\theta)}\,x\,dx = (t_0-t)^2\int_0^{a_j} \mathrm{Re}\,\frac{i\sqrt{1/a_j^2-\theta^2}}{F_j(1/y)}\,y\,dy, \ldots$$

Condition (10), after carrying out the integrations with respect to t and reducing by t_0, is rewritten in the form

$$\frac{2P}{\pi^2 l}\sum_{j=1}^{2}\frac{1}{\rho_j b_j^4}\left[\Phi_0(k)\frac{2n}{2n-1} + \frac{1}{3}B_2(n)\frac{a_j^2 t_0^2}{l^2}\Phi_1(k) + \cdots\right] = v_0 - \frac{hP}{2M}t_0. \tag{11}$$

Here we have used equality (3) and introduced the notation

$$\Phi_0(k) = -\int_0^1 \mathrm{Re}\,\frac{\sqrt{1/\eta^2-1}}{F_j(1/\eta)}\frac{d\eta}{\eta}; \qquad \Phi_1(k) = \int_0^1 \mathrm{Re}\,\frac{\sqrt{1/\eta^2-1}}{F_j(1/\eta)}\,\eta\,d\eta;$$

$$\eta = \frac{y}{a_j}, \qquad k_j = \frac{b_j^2}{a_j^2}.$$

Letting t_0 tend to zero in (11), we obtain the desired additional relation for the case of contact of order n:

$$\frac{4n}{2n-1}\frac{P}{\pi^2 l}\sum_{j=1}^{2}\frac{1}{\rho_j b_j}\Phi_0(k_j)=v_0. \tag{12}$$

For what follows it is convenient to introduce the dimensionless quantities:

$$\bar{\rho}_j=\frac{\rho_j}{\rho_0};\qquad \bar{b}_j=\frac{b_j}{b_0};$$

$$\frac{1}{N_1N_2}=\sum_{j=1}^{2}\frac{\Phi_0(k_j)}{\bar{\rho}_j\bar{b}_j};\qquad \frac{1}{N_2}=\sum_{j=1}^{2}\frac{1}{\bar{\rho}_j\bar{b}_j(1-k_j)};\qquad \psi_n=\frac{(2n-1)!!}{(2n-2)!!}, \tag{13}$$

where ρ_0, b_0 are the characteristic density and velocity of transverse waves of the given system. In the new notation we write

$$P=\psi_n\frac{A_n}{\theta_1+\theta_2}l^{2n};\qquad A_n=\frac{f_1^{(2n)}(0)+f_2^{(2n)}(0)}{(2n)!};\qquad \frac{1}{\theta_1+\theta_2}=2\pi\rho_0b_0^2N_2, \tag{14}$$

and from (3) and (5) we find:

$$l=\left(\frac{\pi}{2}\right)^{\frac{1}{2n-1}}\left(\frac{2n-1}{2n}\right)^{\frac{1}{2n-1}}\psi_1^{-\frac{1}{2n-1}}N_1^{-\frac{1}{2n-1}}A_n^{-\frac{1}{2n-1}}\left(\frac{v_0}{b_0}\right)^{\frac{1}{2n-1}}; \tag{15}$$

$$P=\pi\left(\frac{\pi}{2}\right)^{\frac{2n}{2n-1}}\left(\frac{2n-1}{2n}\right)^{\frac{2n}{2n-1}}\psi_n^{-\frac{1}{2n-1}}N_2N_1^{-\frac{2n}{2n-1}}\rho_0A_n^{-\frac{1}{2n-1}}b_0^{2\frac{n-1}{2n-1}}v_0^{\frac{2n}{2n-1}}; \tag{16}$$

$$t_1=\frac{\psi_n^{\frac{1}{2n-1}}}{\pi\left(\frac{\pi}{2}\right)^{\frac{1}{2n-1}}}\frac{A_n^{\frac{1}{2n-1}}}{N_2N_1^{\frac{2n}{2n-1}}}\frac{M}{\rho_0h}\frac{\left(\frac{2n}{2n-1}\right)^{\frac{2n}{2n-1}}}{b_0^{2\frac{n-1}{2n-1}}v_0^{\frac{1}{2n-1}}}. \tag{17}$$

P is expressed in terms of α_0 by the formula

$$P=\pi^{\frac{2n-1}{n-1}}\left(\frac{\pi}{2}\right)^{\frac{2n}{n-1}}\left(\frac{2n-1}{2n}\right)^{\frac{2n}{n-1}}\psi_n^{-\frac{1}{n-1}}N_1^{\frac{2n}{n-1}}N_2^{\frac{2n-1}{n-1}}\times$$

$$\times b_0^2\left(\frac{2h}{M}\right)^{\frac{n}{n-1}}\rho_0^{\frac{2n-1}{n-1}}A_n^{-\frac{1}{n-1}}\alpha_0^{\frac{n}{n-1}}. \tag{18}$$

Fig. 2

Figure 1: Fig. 2

It follows from this that, as n increases, the relation between P and α_0 tends to a linear one. In general, in the case of contact of order n, we arrive here at the same conclusions as in the three-dimensional problem based on Hertz' s hypothesis: 1) the dependence of t_1 on the initial velocity v_0 weakens as n increases; 2) for $n = \infty$ the pressure force P proves to be proportional to α_0, and the latter is proportional to the ratio of the initial velocity v_0 to b_0. For $n = 1$ formula (18) loses its meaning.

Example. As an example, consider the impact of a circular disk

$$y_1 = f_1(x) = R\left(1 - \sqrt{1 - \frac{x^2}{R^2}}\right) \simeq$$

$$\simeq \frac{1}{2R}x^2 + \frac{1}{8R^3}x^4$$

against the parabola

$$y_2 = f_2(x) = -\frac{1}{2R}x^2$$

(Fig. 2). We take the materials of the bodies to be identical. We have: $n = 2$;

$$k = b^2/a^2; \qquad A_2 = \frac{1}{8R^3}; \qquad \bar{\rho}_j = 1; \qquad \bar{b}_j = 1; \qquad \rho_0 = \rho_j, \quad b_0 = b_j,$$

$$\frac{1}{N_1} = (1-k)\Phi_0; \qquad N_2 = \frac{1-k}{2}; \qquad m_2 = \infty.$$

Formulas (15)–(18) will take the form:

$$l = \sqrt[3]{2\pi N_1 \frac{v_0}{b} R}; \qquad P = \frac{3}{8}\pi^2 N_1 N_2 \sqrt[3]{2\pi N_1 b^2 v_0^4 \rho R};$$

$$t_1 = \frac{8R}{3\pi N_1 N_2 \sqrt[3]{2\pi N_1 b^2 v_0^4}}; \qquad \alpha_0 = \frac{4R}{3\pi N_1 N_2 \sqrt[3]{2\pi N_1}} \left(\frac{v_0}{b}\right)^{2/3}; \tag{19}$$

$$P = \pi^2 \left(\frac{3\pi}{4}\right)^3 \frac{N_1^4 N_2^3}{R} \rho b^2 \alpha_0^2.$$

Table 1

Materials	k	$\Phi_0(k)$
Material with $\lambda = \mu$ (glass)	0.3333	0.8848
Steel	0.2938	0.7659
Incompressible medium	0	0.4346

To compute N_1, one must know

$$\Phi_0(k) = -\int_0^1 \operatorname{Re} \frac{\sqrt{1/\eta^2 - 1}}{F(1/\eta)} \frac{d\eta}{\eta}.$$

This integral at the point $\eta = c/a$ (c is the velocity of Rayleigh surface waves) exists in the sense of the Cauchy principal value. Table 1 gives some values of $\Phi_0(k)$, and Table 2 gives the principal quantities for the impact of a disk on a parabola (both bodies are made of steel; in all cases $R = 1$ cm).

Table 2

Steel ($E = 21 \cdot 10^{11}$ dyn/cm^2, $\rho = 7.8$ g/cm^3, $\nu = 0.2901$, $M = 24.5$ g)

Order of contact	v_0, cm/sec	Ph, dyn	l, cm	t_2, sec	α_0, cm
1	1	18.18	$4 \cdot 10^{-6}$	2.6	0.67
2	1	$0.2 \cdot 10^6$	$3.30 \cdot 10^{-2}$	$250 \cdot 10^{-6}$	$0.7 \cdot 10^{-4}$
2	10	$4.31 \cdot 10^6$	$7.11 \cdot 10^{-2}$	$120 \cdot 10^{-6}$	$2.8 \cdot 10^{-4}$
2	100	$92.83 \cdot 10^6$	$15.32 \cdot 10^{-2}$	$50 \cdot 10^{-6}$	$20.5 \cdot 10^{-4}$
2	200	$233.92 \cdot 10^6$	$19.30 \cdot 10^{-2}$	$40 \cdot 10^{-6}$	$20.5 \cdot 10^{-4}$
2	300	$401.28 \cdot 10^6$	$22.10 \cdot 10^{-2}$	$30 \cdot 10^{-6}$	$20.9 \cdot 10^{-4}$

Let us note that, as is seen from Table 2, the impact time t_2 is greater than the period of the slowest natural vibrations of the disk.

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3. A. Ya. Shtaerman, *Contact Problems of the Theory of Elasticity*, 1949.

Note: Figure translations are in progress. See original paper for figures.