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Abstract

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REFLECTION AND REFRACTION OF AN ELASTIC WAVE OF ARBITRARY FORM IN THE CASE OF A CURVILINEAR INTERFACE

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In applications, the problem of reflection and refraction of nonstationary waves from an arbitrary interface between two elastic media is of interest. If this problem is considered by means of the ray method ([1]), then it naturally splits into two problems: 1) determination of the fields of the reflected and refracted waves in the domain where geometrical optics of elastic waves is applicable, and 2) study of the head (diffracted) waves. In the present note the first problem is considered (in the zeroth approximation of the ray method). In this approximation it turns out that the principle of the isolated element is satisfied. Further, the field of the reflected (refracted) wave is determined at any point. The second problem, in the case of an arbitrary interface, has not yet been solved.

Let two elastic media, characterized by the parameters λ_i, μ_i, and ρ_i, be separated by a surface S of arbitrary form, across which the displacement and stress components remain continuous. An elastic wave is incident on the surface S, its front coinciding with the moving surface Σ_1. Assuming that at every point of S and Σ_1 there exist continuous radii of curvature, it is required to determine the fields of the reflected and refracted waves. The displacement fields of the waves are expediently sought in the neighborhood of the wave fronts (surfaces of discontinuity $t = \tau_\nu(x, y, z)$) in the form

$$\mathbf{u}_\nu = \operatorname{Im} \sum_{n=0}^{\infty} \mathbf{u}_n^\nu f_n(t - \tau_\nu), \tag{1}$$

where

$$\mathbf{u}_n^{(\nu)} = \sum_{k=1}^{3} u_{n,k}^{(\nu)}\, \mathbf{t}_k^{(\nu)}; \tag{2}$$

$$u_{n,k}^{(\nu)} = J_{n,k}^{(\nu)} e^{i\chi_{n,k}{}^{(\nu)}} \quad (k = 1, 2, 3); \tag{3}$$

$$f_n(t-\tau_\nu) = \int_{\omega_0}^{\infty} \frac{e^{i\omega(t-\tau_\nu)}}{[i\omega]^{\,n+1}}\, d\omega \quad (\omega_0 > 1). \tag{4}$$

The quantity $\tau_\nu(x, y, z)$ is the eikonal of the corresponding disturbance. We shall assume that the index $\nu = 1$ corresponds to the prescribed incident wave, while the indices $\nu = 2, 3, \ldots$ refer to the sought fields of the reflected and refracted waves. The quantity $u_{n,k}^{(\nu)}$ will be called the complex amplitude of the n-th approximation in the k-th component. It is convenient to choose the unit vectors $\mathbf{t}_k^{(\nu)}$ $(k = 1, 2, 3)$ so that $\mathbf{t}_1^{(\nu)}$ and $\mathbf{t}_2^{(\nu)}$ lie in the plane tangent

to the surface Σ_ν of the wave front having index ν, and the unit vector $\mathbf{t}_3^{(\nu)}$ was directed along the normal to this surface in the direction of propagation of the wave. From the Lamé equations it follows that, in the general case, $\chi_{n,1}^{(\nu)} \neq \chi_{n,2}^{(\nu)} \neq \chi_{n,3}^{(\nu)}$ $(n \geq 1)$.

We shall be interested in the zeroth approximation. If, in this case, the index ν corresponds to a transverse wave, then, as is known, the wave is polarized in the plane tangent to the corresponding front surface, i.e., one may take $u_{0,3}^{(\nu)} = 0$. In considering reflection processes it is expedient to choose the unit vectors so that $\mathbf{t}_1^{(\nu)}$ lies in the plane of incidence, and $\mathbf{t}_2^{(\nu)}$ in the plane perpendicular to it.

In the case of a longitudinal wave the first term of the series (1) is linearly polarized, i.e., one may put $u_{0,1}^{(\nu)} = u_{0,2}^{(\nu)} = 0$.

If the expansions (1) for the incident, reflected, and refracted waves* are substituted into the boundary conditions and the coefficients standing before the functions $f_n(t-\tau_\nu)$ and their derivatives, having the same character of discontinuity, are equated to one another, then an algebraic system is obtained for the complex intensities $u_{n,k}^{(\nu)}$.

The solutions of the algebraic system for the coefficients standing before the function $f_0'(t-\tau)$, which has the strongest discontinuity, may be written in the form

$$u_{0,k}^{(\nu)}\Big|_{\text{on } S} = V_k^{(\nu)} = u_{0,k}^{(1)} \varkappa_k^{(1\nu)} \qquad (k = 1, 2;\ \nu = 2, \ldots, 5), \tag{5}$$

where $\varkappa_k^{(1\nu)}$ is the corresponding complex coefficient of reflection (refraction) for a plane wave incident on the interface plane at the same angle of incidence α. Formula (5) is applicable only in the illuminated region.

It is seen from expression (5) that, for the zeroth term of (1), the principle of the isolated element is satisfied, i.e., an incident elastic wave is reflected from a curvilinear boundary at each point in the same way as a plane wave would be reflected from a small element of the plane passing through the same point. Usually this principle is postulated ([2]).

In order to determine the principal part of the field of the reflected (refracted) wave at any point M, it is necessary to take into account the geometrical divergence of the corresponding wave, which we shall denote by $\sqrt{d\sigma_0^{(\nu)}/d\sigma^{(\nu)}}$. Then the complex intensity at the point M is determined from

$$u_{0,k}^{(\nu)} = V_k^{(\nu)} \sqrt{\frac{d\sigma_0^{(\nu)}}{d\sigma^{(\nu)}}}. \tag{6}$$

It is easy to see that in a homogeneous isotropic medium

$$\sqrt{\frac{d\sigma_0^{(\nu)}}{d\sigma^{(\nu)}}} = \sqrt{\frac{r_1^{(\nu)} r_2^{(\nu)}}{\left(r_1^{(\nu)} + l\right)\left(r_2^{(\nu)} + l\right)}}, \tag{7}$$

where $r_1^{(\nu)}, r_2^{(\nu)}$ are the principal radii of curvature of the front surface of the reflected (refracted) wave at the point of incidence A; l is the distance along the ray from the point A to the point M.

If $r_1^{(\nu)}, r_2^{(\nu)}$ are determined by means of simple geometrical considerations** and formula (7) is used, then for the geometrical divergence–

* For angles of incidence α greater than the critical angle, it is necessary to take into account the fields of screened (surface) waves, which have discontinuities only at the boundary S.

** Explicit expressions for $r_i^{(\nu)}$ are not given here.

tion of the reflected (refracted) wave one obtains the expression

$$\sqrt{\frac{d\sigma_0^{(\nu)}}{d\sigma^{(\nu)}}} = \frac{\cos\beta_\nu}{\sqrt{\cos^2\beta_\nu + (A_\nu + B_\nu\cos^2\beta_\nu)\, l + (A_\nu B_\nu - C_\nu^2)\, l^2}}, \tag{8}$$

in which

$$A_\nu = \frac{m_\nu \cos^2\alpha}{r_{11}^{(1)}} + \frac{m_\nu\cos\alpha \pm \cos\beta_\nu}{R_{11}},$$

$$B_\nu = \frac{m_\nu}{r_*^{(1)}} + \frac{m_\nu\cos\alpha \pm \cos\beta_\nu}{R_*}, \tag{9}$$

$$C_\nu = \mp\left[\frac{m_\nu\cos\alpha\sin 2\psi^{(1)}}{2}\left(\frac{1}{r_1^{(1)}} - \frac{1}{r_2^{(1)}}\right) - \frac{(m_\nu\cos\alpha \pm \cos\beta_\nu)\sin 2\Phi}{2}\left(\frac{2}{R_1} - \frac{1}{R_2}\right)\right],$$

$$m_\nu = \frac{v_\nu}{v_1}.$$

In formulas (9) the following notation has been introduced: β_ν is the angle of reflection (refraction); R_1, R_2 and $r_1^{(1)}, r_2^{(1)}$ are the principal radii of curvature of the surfaces S and Σ_1; R_{11} and $r_{11}^{(1)}$ are the radii of curvature of the normal sections of the surfaces S and Σ_1 in the plane of incidence; R_* and $r_*^{(1)}$ are the radii of curvature of the normal sections of the surfaces S and Σ_1 in the plane perpendicular to the plane of incidence; Φ and $\psi^{(1)}$ are the angles formed between the plane of incidence and the principal planes of the surfaces S and Σ_1.

The normal to the surface S forms an obtuse angle with the unit vector $\mathbf{t}_3^{(1)}$. The radii of curvature of the normal sections of the surfaces Σ_ν and S are considered positive if the unit normals corresponding to these surfaces are directed toward the centers of curvature of the normal sections. All these quantities are determined at the point of incidence. In formulas (9), the upper sign refers to the reflected wave, and the lower sign to the refracted wave.

The geometrical divergence becomes infinite at focusing points (when $l = -r_i^{(\nu)}$) and becomes purely imaginary when $r_{k_1}^{(\nu)} < l < r_{k_2}^{(\nu)}$. It follows from formulas (1)–(4) that the complexity of the geometrical divergence leads to a change in the type of discontinuity.

In conclusion we note that formula (8) passes into the Riblet–Barker formula ([3]) (for $m = 1$, $r_1^{(1)} = r_2^{(1)}$). In certain special cases it can be shown to be identical with the tensor formula of V. A. Fock ([4]). The author considers it a pleasant duty to express gratitude to G. I. Petrashen for a number of valuable suggestions.

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Note: Figure translations are in progress. See original paper for figures.