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Abstract

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MATHEMATICS

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ON THE REGULARITY OF CONVEX SURFACES WITH A REGULAR METRIC IN SPACES OF CONSTANT CURVATURE

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Let, in a space of constant curvature K, we have a convex surface F (a domain on the boundary of a convex body). We shall say that the metric of the surface is regular (k times differentiable, analytic) if on the surface a coordinate net (u) can be introduced such that the coefficients $\tilde{g}_{ij}$ of its line element

$$d\tilde{s}^2 = \tilde{g}_{ij} du^i du^j \quad (i, j = 1, 2)$$

are regular (respectively, k times differentiable, analytic) functions of the variables u^i.

Introduce in the space an analytic coordinate net (v); this means that the coefficients g_{ij} of the line element of the space

$$ds^2 = g_{ij} dv^i dv^j$$

are analytic functions of v^1, v^2, v^3. In this case the spatial coordinates v^i of a point of the surface will be defined functions of its intrinsic coordinates u^i:

$$v^i = v^i(u^1, u^2).$$

The present note is devoted to the following question. To what extent does the intrinsic regularity of a surface (the regularity of $\tilde{g}_{ij}(u^1, u^2)$) entail the regularity of its external form (the regularity of the functions $v^i(u^1, u^2)$)? This question arises in connection with the application of the synthetic methods of the theory of convex surfaces (^{1}) to various problems of bending of regular surfaces in spaces of constant curvature. In the case of Euclidean space ($K = 0$) the question was solved in the author’ s work (^{2}). The method used there can in principle also be applied in the general case ($K \neq 0$). The final result obtained in this way consists in the following theorem.

Theorem. *If the metric of a convex surface in a space of constant curvature is k times differentiable ($k \geq 4$), and the Gaussian curvature of the surface is positive and greater than the curvature of the space, then the surface is at least $k-1$ times differentiable. If the metric of the surface is analytic, then the surface is analytic.*

In other words, if the coefficients $\tilde{g}_{ij}(u^1, u^2)$ of the line element of the surface are k times differentiable ($k \geq 4$), and the Gaussian curvature expressed through $\tilde{g}_{ij}$ and their derivatives is positive and greater than the curvature of the space, then the spatial coordinates of a point of the surface $v^i(u^1, u^2)$, as functions of the coordinates u^1, u^2 on the surface, are $k-1$ times differentiable. If the functions $\tilde{g}_{ij}$ are analytic, then the functions $v^i(u^1, u^2)$ are also analytic.

The requirement that the Gaussian curvature of the surface be not merely greater than the curvature of the space, but also positive, in the case of a space of negative curvature (Lobachevsky space) seems unnatural. Unfortunately, only under this condition is it possible to obtain a priori estimates for the normal curvatures of the surface, to which one has to appeal at two essential points in the proof of the theorem.

We shall give a description of the main stages of the proof. For definiteness we shall assume the curvature of the space to be negative. As for the metric of the surface, for simplicity we shall assume it to be analytic.

Let X_0 be an arbitrary point of the surface, in a neighborhood of which it is required to establish the analyticity of the surface. Cut off from the surface by a plane a small cap ω containing the point X_0. Approximate the boundary γ of the cap by an internal analytic curve $\tilde{\gamma}$ with positive geodesic curvature. Denote by $\tilde{\omega}$ the domain on the surface bounded by the curve $\tilde{\gamma}$.

The proof of the theorem is reduced to proving the following three assertions:

1. There exists an analytic cap isometric to $\tilde{\omega}$.
2. There exists an analytic cap isometric to ω.
3. The cap isometric to ω is equal to ω.

The proof of the first assertion is analytically reduced to proving the existence of a solution of the bending equation, which, as in the case of Euclidean space, with an appropriate choice of a coordinate system in the space, is a Monge–Ampère equation of elliptic type. As such a coordinate system it is convenient to take a semigeodesic system, taking as coordinate surfaces $z = \text{const}$ the surfaces equidistant from the plane of the base of the cap. Moreover, if on the surface one also introduces semigeodesic coordinates u, v, then the bending equation in the case of a space of curvature $K = -1$ takes the form

$$(r - \operatorname{th} z \cdot (1 - p^2)) \left(t - \operatorname{th} z(c^2 - q^2) + cc_u p - \frac{c_v}{c} q\right) - \left(s + \operatorname{th} z \cdot pq - \frac{c_u}{c} q\right)^2 = (\tilde{k}+1)(c^2 - c^2 p^2 - q^2),$$

where p, q, r, s, t are the derivatives of z with respect to u and v; c^2 is the coefficient in the linear element of the surface $ds^2 = du^2 + c^2 dv^2$, and $\tilde{k}$ is the Gaussian curvature.

For the solvability of the boundary-value problem for this equation, according to the general theorem of S. N. Bernstein, it is enough to be able a priori to establish estimates for the assumed solution and its derivatives of the first two orders. Such estimates can indeed be obtained by a slight modification of the methods used in [2].

The proof of the second assertion, in its essential part, rests on the possibility of indicating estimates of the normal curvature of the cap, whose existence is asserted in item 1, depending only on quantities determined by the intrinsic metric of the cap and by the distance of the point from its base. These estimates are obtained by the usual consideration of the auxiliary function $\varkappa z$ [2], where $\varkappa$ is the normal curvature, and z is the distance of the point from the base of the cap.

As for the third assertion, its proof can be constructed on considerations similar to those used in [3] for caps in Euclidean space. It is precisely in this way that the equality of isometric caps in Lobachevsky space was proved by I. Ya. Danelich [4].

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