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Abstract

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ON THE QUESTION OF THE SUPERFLUIDITY CONDITION IN THE THEORY OF NUCLEAR MATTER

Having in mind applications to the theory of nuclear matter, let us consider a model dynamical system with Hamiltonian

$$H = \sum_{k,\sigma} \{E(k) - E_F\} a^+_{k\sigma} a_{k\sigma} + \\ + \frac{1}{2V} \sum_{(k,k',\ldots,\sigma\ldots)} J(k,k' \mid \sigma_1,\sigma_2,\sigma_2',\sigma_1') a^+_{k\sigma_1} a^+_{-k\sigma_2} a_{-k'\sigma_2'} a_{k'\sigma_1'}. \tag{1}$$

Here σ is a discrete index characterizing, for example, spin and isotopic spin; E_F, the "Fermi energy," is a parameter playing the role of the chemical potential; V is the volume of the system.

The incompleteness, the "model" character of such a Hamiltonian is due to the fact that it takes into account only interactions of pairs of particles with opposite momenta. For convenience of notation it is expedient to introduce, instead of the momentum index k, the index q of the pair $(k, -k)$; q and $-q$ describe one and the same pair; summation over q is understood as summation over different pairs. In this case, obviously, it will be necessary to introduce one more additional index $\rho = \pm 1$ and to describe k as (q, ρ). It is expedient to combine ρ, as a discrete index, with σ and to put $s = (\sigma, \rho)$. In these notations the Hamiltonian (1) under consideration is represented in the form

$$H = \sum_{q,s} \{E(q) - E_F\} a^+_{qs} a_{qs} + \\ + \frac{1}{2V} \sum_{(q,q',\ldots,s\ldots)} I(q,q' \mid s_1,s_2,s_2',s_1') a^+_{qs_1} a^+_{qs_2} a_{q's_2'} a_{q's_1'}. \tag{2}$$

We shall show that the ground state in our scheme can be found asymptotically exactly, for the process of taking the limit $V \to \infty$.

Here it will be convenient to use a variant of the device of note ([1]), proposed by D. N. Zubarev and Yu. A. Tserkovnikov. Let us introduce certain c-functions $A_q(s_1, s_2)$ and write the Hamiltonian (2) in the form

$$H = U_0 + H_0 + H_1,$$

where

$$U_0 = \text{const} = -\frac{1}{2V} \sum I(q, q' \mid s_1, s_2, s_2', s_1') A_q^*(s_1, s_2) A_{q'}(s_1', s_2'),$$

$$H_0 = \sum_q H_q; \qquad H_1 = \frac{1}{2V} \sum I(q, q' \mid s_1, s_2, s_2', s_1') B_q^+(s_1, s_2) B_{q'}(s_1', s_2'),$$

where

$$H_q = \{E(q) - E_F\} \sum_s a_{qs}^+ a_{qs} + \frac{1}{2V} \sum \{I(q, q' \mid s_1, s_2, s_2', s_1') A_{q'}(s_1', s_2')\, a_{qs_1}^+ a_{qs_2}^+ + I(q', q \mid s_1, s_2, s_2', s_1') A_{q'}^*(s_1, s_2)\, a_{qs_2'} a_{qs_1'}\}, \tag{3}$$

$$B_q(s_1, s_2) = a_{qs_2} a_{qs_1} - A_q(s_1, s_2).$$

Since H_q is a quadratic form in Fermi operators, its diagonalization is carried out elementarily, by means of the linear canonical transformation

$$a_{qs} = \sum_{s'} \{u(q, s, s') \alpha_{qs'} + v(q, s, s') \alpha_{qs'}^+\}. \tag{4}$$

The functions u, v entering here must satisfy the orthonormality relations

$$\xi \equiv \sum_{s''} \{u^*(q, s, s'') u(q, s', s'') + v^*(q, s, s'') v(q, s', s'')\} = \delta_{s,s'},$$

$$\eta \equiv \sum_{s''} \{u(q, s, s'') v(q, s', s'') + v(q, s, s'') u(q, s', s'')\} = 0. \tag{5}$$

Having determined u, v from the secular equations corresponding to the form (3), we bring it to the form:

$$H_q = \Gamma_q + \sum_s \varepsilon_s(q)\alpha_{qs}^+\alpha_{qs}.$$

Therefore, for the Hamiltonian H_0, the ground state C_0 will be the vacuum state for the new fermion amplitudes

$$\alpha_{ks}C_0 = 0.$$

We now choose the c-functions A in such a way that

$$\langle C_0^* B_q(s_1, s_2) C_0 \rangle = 0,$$

and take into account the important fact that H_q, B_q, B_q^+, corresponding to different q, all commute with one another.

Then, with the aid of the reasoning given in (1), it is not difficult to show that the contribution to the ground-state energy arising from H_1 becomes negligibly small in comparison with the contribution from $U_0 + H_0$ as $V \to \infty$. Roughly speaking, this proposition is due to the circumstance that $\overline{H_1^2}$ remains finite as $V \to \infty$, whereas the energy is proportional to V.

Thus, by a suitable choice of the functions u, v, one can ensure that the mean value $\overline{H} = \langle C_0^* H C_0 \rangle$ represents, asymptotically exactly, the ground-state energy for the Hamiltonian H under consideration.

It follows from this that the actual determination of u, v can be carried out in the following way: we substitute the transformation formulas (4) into the expression $\overline{H}$ and find

$$\begin{aligned} \overline{H} = & \sum_{q,s}\{E(q) - E_F\}\sum_{s'} v^*(q,s,s')v(q,s,s') + \\ & + \frac{1}{2V}\sum_{(q,q',\ldots s\ldots)} I(q,q' \mid s_1,s_2,s_2',s_1')\sum_s v^*(q,s_1,s)u^*(q,s_2,s)\times \\ & \times \sum_s u(q,s_2',s)v(q,s_1',s) = \mathcal{E}(u,v). \end{aligned} \tag{6}$$

Then u, v must be determined by the condition of the minimum of the form $\mathcal{E}(u,v)$ in the presence of the additional conditions (5). For such u, v the expression $\mathcal{E}$ gives the required value of the ground-state energy.

The corresponding stationarity equation will be

$$\delta\tilde{\mathcal{E}} = \delta\left\{\mathcal{E} + \sum_{q,s,s'}\left(\lambda(q,s,s')\,\xi(q,s,s') + \mu(q,s,s')\,\eta(q,s,s') + \right.\right.$$

$$+\mu^*(q,s,s')\,\eta^*(q,s,s'))\} = 0, \tag{7}$$

where λ, μ are Euler multipliers. It is easy to note that this equation always admits the trivial solution

$$u_q = \theta_G(q)\delta_{s,s'}, \qquad v_q = \theta_F(q)\delta_{s,s'},$$

$$\mu = 0, \qquad \lambda = \theta_F(q)(E_F - E(q))\delta_{s,s'}, \tag{8}$$

where $\theta_F(q)$ is equal to unity inside the Fermi sphere and to zero outside; $\theta_G(q) = 1 - \theta_F(q)$.

As is seen, in the corresponding state $C_0^{(n)}$ the interaction is ineffective, and its entire contribution to the energy is made only by the first term of the Hamiltonian (1).

To decide the question of when the energy $C_0^{(n)}$ will not be minimal and when, consequently, the ground state $C_0^{(s)}$ will be characterized by a nontrivial solution of equations (7), let us turn to the known procedure of variational calculus. Let us construct the expression for the second variation $\delta^2\tilde{\mathcal{E}}$ for the trivial solution. We find:

$$\delta^2\tilde{\mathcal{E}} = \sum_{q,s,s'} |E(q) - E_F|\,\Psi^*(q,s,s')\Psi(q,s,s') +$$

$$+\frac{1}{2V}\sum_{(q,q',\ldots s\ldots)} I(q,q' \mid s_1,s_2,s_2',s_1')\Psi^*(q,s_1,s_2)\Psi(q',s_1',s_2'),$$

where

$$\Psi(q,s,s') = \theta_F(q)\delta u(q,s,s') - \theta_G(q)\delta v(q,s,s');$$

the functions Ψ are connected only by the antisymmetry conditions: $\Psi(q,s',s) = -\Psi(q,s,s')$, obtained upon variation of the orthonormality conditions.

Let us now return to the system of indices adopted in writing the Hamiltonian (1). We obtain:

$$\delta^2\tilde{\mathcal{E}} = \sum_{k,\sigma,\sigma'} |E(k) - E_F|\,\Psi^*(k,\sigma,\sigma')\Psi(k,\sigma,\sigma') +$$

$$+\frac{1}{2V}\sum_{(h,k',\ldots\sigma\ldots)} J(k,k' \mid \sigma_1,\sigma_2,\sigma_2',\sigma_1')\Psi^*(k,\sigma_1,\sigma_2)\Psi(k',\sigma_1',\sigma_2').$$

The antisymmetry condition will be

$$\Psi(-k,\sigma_2,\sigma_1) = -\Psi(k,\sigma_1,\sigma_2).$$

As is seen, the sign of $\delta^2\tilde{\mathcal{E}}$ can be made negative if and only if the equation

$$2\,|E(k)-E_F|\,\Psi(k,\sigma_1,\sigma_2)+$$

$$+\frac{1}{V}\sum_{(k',\sigma_1',\sigma_2')} J(k,k' \mid \sigma_1,\sigma_2,\sigma_2',\sigma_1')\Psi(k',\sigma_1',\sigma_2') = E\Psi(k,\sigma_1,\sigma_2) \qquad (9)$$

has an eigen-solution with a negative value of E.

In this case the energy $C_0^{(n)}$ will not be minimal, and a ground state $C_0^{(s)}$ of another type arises, characterized by a nontrivial solution of equations (7).

It is interesting to note that equation (9), written in the r-representation (for an interaction not depending on velocity),

$$2\,|E(k)-E_F|\,\Psi(\mathbf{r},\sigma_1,\sigma_2) + \sum_{(\sigma_1',\sigma_2')} \Phi(\mathbf{r} \mid \sigma_1,\sigma_2,\sigma_2',\sigma_1')\Psi(\mathbf{r},\sigma_1',\sigma_2') =$$

$$= E\Psi(\mathbf{r},\sigma_1,\sigma_2), \qquad (10)$$

very much resembles the Schrödinger equation for the two-body problem in the center-of-inertia system. The difference lies in the peculiar form of the "kinetic-energy" operator. This difference naturally disappears in the case of zero density, when $E_F = 0$.

The equation (10) obtained can be applied to the investigation of the question of the superfluidity of nuclear matter as a criterion for the instability of the normal state.

In conclusion, let us note that the arguments presented are generalized for calculating the free energy at a temperature different from zero. Here rather complicated nonlinear equations are obtained, but the equations for determining the critical temperature of the phase transition will again be linear. Thus, for example, if we restrict ourselves to the case of a system of particles of one kind, interacting only with opposite spins, then from equation (12) of work ([1]) we find

$$2\,|E(k) - E_F|\,\mathrm{cth}\,\frac{|E(k) - E_F|}{2\theta}\Psi(\mathbf{r}) + \Phi(\mathbf{r})\Psi(\mathbf{r}) = 0,$$

where θ is the critical temperature.

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CITED LITERATURE

[1] N. N. Bogolyubov, D. N. Zubarev, Yu. A. Tserkovnikov, DAN, **117**, No. 5 (1957).

Note: Figure translations are in progress. See original paper for figures.