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Abstract

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MATHEMATICS

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A FURTHER STRENGTHENING OF JACKSON'S THEOREM ON APPROXIMATION OF CONTINUOUS FUNCTIONS BY ORDINARY POLYNOMIALS

(Presented by Academician M. A. Lavrentiev on 18 III 1958)

S. M. Nikol'skii was the first to discover ⁽¹⁾ that, for functions of the class Lip 1, the well-known Jackson theorem on approximation of continuous functions by ordinary polynomials admits a strengthening. A. F. Timan strengthened and generalized S. M. Nikol'skii's result first to functions of the classes Lip(r, α), $r = 0, 1, 2, \dots$, $0 < \alpha \leq 1$ ⁽²⁾, and then to arbitrary functions having a continuous r -th derivative ($r \geq 0$) ⁽³⁾. In the present note we shall give a further strengthening of Jackson's theorem.

We shall call the function

$$\omega_2(\delta; f) = \omega_2(\delta) = \sup_{|x'' - x'| \leq \delta} \left| f(x') - 2f\left(\frac{x' + x''}{2}\right) + f(x'') \right|, \quad x', x'' \in [a, b]. \quad (1)$$

the *modulus of smoothness* of a continuous function $f(x)$ given on some segment $[a, b]$.

We shall prove a theorem from which A. F. Timan's theorem ⁽³⁾ follows as a consequence and which strengthens Timan's theorem in all cases when the modulus of smoothness of the r -th derivative ($r \geq 0$) of the given function is infinitely small of higher order than the modulus of continuity of this derivative. The theorem obtained, as is seen from ^(4, 5), cannot be strengthened either for functions of the classes Lip($r; \alpha$), or for functions having an r -th derivative ($r \geq 0$) that is quasismooth. For its proof we shall use the approximation process constructed by us in ⁽⁵⁾ for the approximation of quasismooth functions (i.e., functions $f(x)$ for which $\omega_2(\delta; f) = O(\delta)$).

Theorem. *If a function $f(x)$, given on the segment $[a, b]$, has there an r -th (r an integer ≥ 0) continuous derivative $f^{(r)}(x)$, then for this function, for*

any $n = 1, 2, \dots$, one can construct an ordinary polynomial $P_n(x)$ of degree not exceeding n such that, for every $x \in [a, b]$, the inequality

$$|f(x) - P_n(x)| \leq \frac{C}{n^r} \left(\sqrt{(b-x)(x-a)} + \frac{1}{n} \right)^r \times \left[\omega_2^{(r)} \left(\frac{\sqrt{(b-x)(x-a)}}{n} \right) + \omega_2^{(r)} \left(\frac{1}{n^2} \right) \right], \quad (2)$$

holds, where C is a constant independent of n ; $\omega_2^{(r)}(\delta)$ is the modulus of smoothness of the derivative $f^{(r)}(x)$:

$$\omega_2^{(r)}(\delta) = \sup_{|x''-x'| \leq \delta} \left| f^{(r)}(x') - 2f^{(r)} \left(\frac{x'+x''}{2} \right) + f^{(r)}(x'') \right|, \quad x', x'' \in [a, b]. \quad (3)$$

For the proof of this theorem we shall need the following facts:

A. If a continuous function $f(x)$, defined on the interval $[0, 1]$, has modulus of smoothness $\omega_2(f; \delta)$, then the modulus of smoothness of the function $F(x)$, obtained by odd continuation of $f(x)$ to the segment $[-1, 1]$, will satisfy the inequality $\omega_2(F; \delta) \leq 5\omega_2(f; \delta)$, $\delta \in [0, 1]$.

B. For every continuous function $f(x)$, for every $\lambda > 0$, the inequality $\omega_2(f; \lambda\delta) \leq (\lambda + 1)^2 \omega_2(f; \delta)$ holds.

C. For each integer $r \geq 0$ there exists a unique pair of polynomials $\pi_r^{(0)}(x)$ and $\pi_r^{(1)}(x)$ of degree $\leq 2r + 1$ and of the form

$$\begin{aligned} \pi_r^{(0)}(x) &= x^{r+1}(a_0x^r + a_1x^{r-1} + \dots + a_{r-1}x + a_r), \\ \pi_r^{(1)}(x) &= (1-x)^{r+1}(b_0x^r + b_1x^{r-1} + \dots + b_{r-1}x + b_r) \end{aligned}$$

such that

$$\pi_r^{(0)}(x) + \pi_r^{(1)}(x) \equiv 1.$$

D. For arbitrary natural k and n , the expressions

$$D_{nk}(x) = \frac{1}{\gamma_{nk}} \left(\frac{\sin^2 \frac{1}{2} n \arccos(1 - \frac{1}{2} x^2)}{\sin^2 \frac{1}{2} \arccos(1 - \frac{1}{2} x^2)} \right)^{2k}, \quad x \in [-\sqrt{2}, \sqrt{2}],$$

where γ_{nk} is such a number that

$$\int_{-1}^1 D_{nk}(x) dx = 1,$$

are positive even polynomials of degree $2k(n-1)$ with the following properties:

$$\text{a) } \int_{-1}^1 D_{nk}(x) dx = 1; \quad \text{b) } \int_{-\sqrt{2}}^{\sqrt{2}} D_{nk}(x)|x|^i dx = O\left(\frac{1}{n^i}\right), \quad 0 < i \leq 2k-2;$$

c) for arbitrary fixed δ satisfying $0 < \delta < 1$,

$$\int_{\delta}^{\sqrt{2}} D_{nk}(x) dx = O\left[\left(\frac{1}{n\delta}\right)^{2k-1}\right].$$

Assertion A is proved exactly as Lemma 1 in ⁽⁵⁾; assertion B was proved by Marchaud; assertions C and D were proved in ⁽⁵⁾ (see § 2).

Proof of the theorem. Without loss of generality we shall assume that: a) the function $f(x)$ is given on the segment $[0, 1]$; b) $f^{(r)}(0) = f^{(r)}(1) = 0$.

1°. We first consider the case $r = 0$. Introduce the auxiliary function $g(x) = f(1-x)$ and, using A, extend each of the functions $f(x)$ and $g(x)$ to the segment $[0, 4]$ so that the moduli of smoothness of the functions obtained do not exceed, respectively, $A\omega_2(f; \delta)$ and $A\omega_2(g; \delta) = A\omega_2(f; \delta)$, where A is a constant. The functions obtained after extension will still be denoted by $f(x)$ and $g(x)$. Starting from the function $f(x)$, we construct two auxiliary functions

$$\varphi(x) = 2 \int_0^{1/3} f(x^2 + 9u^2) D_{nk}(u) du; \quad \psi(x) = 2 \int_0^{1/3} f(x^2 + \frac{9}{2}u^2) D_{nk}(u) du, \quad (4)$$

where k is a fixed number ≥ 3 , and construct two even polynomials of degree $\leq 2k(n-1)$:

$$P_1(x^2) = \frac{1}{6} \int_{-2}^2 f(u^2) \left[D_{nk}\left(\frac{u+x}{3}\right) + D_{nk}\left(\frac{u-x}{3}\right) \right] du, \quad (5)$$

$$P_2(x^2) = \frac{\sqrt{2}}{6} \int_{-2}^2 f(u^2) \left[D_{nk}\left(\frac{u+x}{3}\sqrt{2}\right) + D_{nk}\left(\frac{u-x}{3}\sqrt{2}\right) \right] du. \quad (6)$$

Then, taking into account that, by virtue of B,

$$\omega_2(u^2) = \omega_2\left(n^2 u^2 \frac{1}{n^2}\right) \leq (n^2 u^2 + 1)^2 \omega_2\left(\frac{1}{n^2}\right),$$

$$\omega_2(xu) \leq (nu + 1)^2 \omega_2\left(\frac{x}{n}\right),$$

we, taking D into consideration, obtain:

$$\begin{aligned}
 |f(x^2) - 2\psi(x) + \varphi(x)| &= \left| \int_{-1/3}^{1/3} [f(x^2) - 2f(x^2 + \frac{9}{2}u^2) + f(x^2 + 9u^2)] D_{nk}(u) du \right. \\
 &\quad \left. + \int_{-1}^{-1/3} + \int_{1/3}^1 f(x^2) D_{nk}(u) du \right| \\
 &\leq A\omega_2\left(\frac{1}{n^2}\right) \int_{-1/3}^{1/3} (n^2u^2 + 1)^2 D_{nk}(u) du + O\left(\frac{1}{n^{2k-1}}\right) \\
 &= O\left[\omega_2\left(\frac{1}{n^2}\right)\right]; \tag{7}
 \end{aligned}$$

$$\begin{aligned}
 |-\varphi(x) + P_1(x^2)| &\leq \left| \int_0^{1/3} [f(x^2 + 6xu + 9u^2) - 2f(x^2 + 9u^2) + \right. \\
 &\quad \left. + f(x^2 - 6xu + 9u^2)] D_{nk}(u) du + O\left(\frac{1}{n^{2k-1}}\right) \right| \leq \\
 &\leq A\omega_2\left(\frac{x}{n}\right) \int_0^{1/3} (nu + 1)^2 D_{nk}(u) du + O\left(\frac{1}{n^{2k-1}}\right) = O\left[\omega_2\left(\frac{x}{n}\right)\right]. \tag{8}
 \end{aligned}$$

and similarly

$$|-\psi(x) + P_2(x^2)| = O\left[\omega_2\left(\frac{x}{n}\right)\right]. \tag{8'}$$

By virtue of the inequalities obtained, we have:

$$\begin{aligned}
 |f(x^2) - 2P_2(x^2) + P_1(x^2)| &\leq |f(x^2) - 2\psi(x) + \varphi(x)| + \\
 &+ 2|\psi(x) - P_2(x^2)| + |-\varphi(x) + P_1(x^2)| \leq L \left[\omega_2\left(\frac{x}{n}\right) + \omega_2\left(\frac{1}{n^2}\right) \right], \tag{9}
 \end{aligned}$$

where L is a constant. Therefore, denoting by $P_f(x)$ the polynomial of degree $\leq 2k(n-1)$: $P_f(x^2) = 2P_2(x^2) - P_1(x^2)$, for all $x \in [0, 1]$ we shall have

$$|f(x) - P_f(x)| \leq L \left[\omega_2\left(\frac{\sqrt{x}}{n}\right) + \omega_2\left(\frac{1}{n^2}\right) \right]. \tag{10}$$

Analogously, for $g(x)$ we find a polynomial $P_g(x)$ of degree $\leq 2k(n-1)$ such that

$$|f(x) - P_g(1-x)| = |g(1-x) - P_g(1-x)| \leq L \left[\omega_2 \left(\frac{\sqrt{1-x}}{n} \right) + \omega_2 \left(\frac{1}{n^2} \right) \right]. \quad (10')$$

Therefore, denoting by $P(x)$ the polynomial of degree $\leq k(n-1) + 1 = n_1$: $P(x) = (1-x)P_f(x) + xP_g(1-x)$, for all $x \in [0, 1]$, by virtue of (10) and (10') we shall have:

$$\begin{aligned} |f(x) - P(x)| &= (1-x)|f(x) - P_f(x)| + x|f(x) - P_g(1-x)| \leq \\ &\leq L \left[(1-x)\omega_2 \left(\frac{\sqrt{x}}{n} \right) + x\omega_2 \left(\frac{\sqrt{1-x}}{n} \right) + \omega_2 \left(\frac{1}{n^2} \right) \right] \leq \\ &\leq L_1 \left[\omega_2 \left(\frac{\sqrt{x(1-x)}}{n_1} \right) + \omega_2 \left(\frac{1}{n_1^2} \right) \right], \end{aligned}$$

where L_1 is a constant, and the theorem for the case $r = 0$ is proved.

2°. Suppose the theorem holds for $i = r-1$; we shall prove its validity for $i = r$. Extend $f(x)$ to the segment $[0, 4]$ so that the modulus of smoothness of its r -th derivative $f^{(r)}(x)$ increases by no more than 5^2 times. Since the theorem is true for $i = r-1$, for each $n = 1, 2, \dots$ there exists a polynomial $\bar{P}_n(x)$ of degree $\leq n$ such that, for all $x \in [0, 4]$,

$$|f^{(r)}(x) - \bar{P}_n(x)| \leq \frac{C}{n^{r-1}} \left(\sqrt{x(4-x)} + \frac{1}{n} \right)^{r-1} \left[\omega_2^{(r)} \left(\frac{\sqrt{x(4-x)}}{n} \right) + \omega_2^{(r)} \left(\frac{1}{n^2} \right) \right], \quad (11)$$

where C is a constant. Setting $V_n(x) = \int_0^x \bar{P}_n(x) dx$, $f_1(x) = f(x) - V_n(x)$, and, having fixed some $k \geq r+2$, form the even polynomial $\bar{P}_0(x^2)$:

$$\bar{P}_0(x^2) = \frac{1}{3} \int_{-1}^1 f_1(4u^2) \left\{ D_{nk} \left(\frac{2u+x}{3} \right) + D_{nk} \left(\frac{2u-x}{3} \right) \right\} du.$$

Then, if we put $P_0(x) = \bar{P}_0(x) + V_n(x)$, then for all $x \in [0, 1]$, by virtue of (2), we shall have (taking into account that $\omega_2^{(r)} \left(\frac{x+3u}{n} \right) \leq 9\omega_2^{(r)} \left(\frac{x}{n} \right) + 9(3nu + 1)^2 \omega_2^{(r)} \left(\frac{1}{n^2} \right)$):

$$\begin{aligned}
 |P_0(x^2) - f(x^2)| &= |\bar{P}_0(x^2) - f_1(x^2)| \\
 &= \left| \frac{1}{2} \int_{(-2-x)/3}^{(2-x)/3} \{f_1[(x+3u)^2] - f_1(x^2)\} D_{nk}(u) du \right. \\
 &\quad \left. + \frac{1}{2} \int_{(-2+x)/3}^{(2+x)/3} \{f_1[(x-3u)^2] - f_1(x^2)\} D_{nk}(u) du + O\left(\frac{1}{n^{2k-1}}\right) \right| \\
 &\leq 2 \int_0^{1/3} (6xu + 9u^2) |f_1'[(x \pm 30u)^2]| D_{nk}(u) du + O\left(\frac{1}{n^{2k-1}}\right) \\
 &\leq \frac{CC_1}{n^{r-1}} \int_0^1 (6xu + 9u^2) \left(x + 3u + \frac{1}{n}\right)^{r-1} \left[\omega_2^{(r)}\left(\frac{x+3u}{n}\right) + \omega_2'\left(\frac{1}{n^2}\right)\right] D_{nk}(u) du \\
 &\quad + O\left(\frac{1}{n^{2k-1}}\right) \\
 &\leq \frac{C_2}{n^r} \left(x + \frac{1}{n}\right)^r \left[\omega_2^{(r)}\left(\frac{x}{n}\right) + \omega_2^{(r)}\left(\frac{2}{n^2}\right)\right], \tag{12}
 \end{aligned}$$

where C_1 and C_2 are constants. Similarly, for the function $g(x) = f(1-x)$ we find a polynomial $P_1(x)$ of degree $\leq k(n-1)$ such that

$$\begin{aligned}
 |P_1[(1-x)^2] - f(x^2)| &= |P_1[(1-x)^2] - g[(1-x)^2]| \\
 &\leq \frac{C_2}{n^r} \left(1-x + \frac{1}{n}\right)^r \left[\omega_2^{(r)}\left(\frac{1-x}{n}\right) + \omega_2^{(r)}\left(\frac{1}{n^2}\right)\right]. \tag{12'}
 \end{aligned}$$

Therefore, denoting by $P(x)$ the polynomial of degree $\leq kn + r = n_1$:

$$P(x) = \pi_r^{(1)}(x)P_0(x) + \pi_r^{(0)}(x)P_1(1-x),$$

by virtue of (6) and (7), for all $x \in [0, 1]$

$$\begin{aligned}
 |f(x) - P(x)| &\leq |\pi_r^{(0)}(x)| |f(x) - P_1(1-x)| + |\pi_r^{(1)}(x)| |f(x) - P_0(x)| \\
 &\leq \frac{C_2}{n^r} \left\{ M_1 x^{r+1} \left(\sqrt{1-x} + \frac{1}{n}\right)^r \left[\omega_2^{(r)}\left(\frac{\sqrt{1-x}}{n}\right) + \omega_2^{(r)}\left(\frac{1}{n^2}\right)\right] \right. \\
 &\quad \left. + M_2 (1-x)^{r+1} \left(\sqrt{x} + \frac{1}{n}\right)^r \left[\omega_2^{(r)}\left(\frac{\sqrt{x}}{n}\right) + \omega_2^{(r)}\left(\frac{1}{n^2}\right)\right] \right\} \\
 &\leq \frac{M}{n_1^r} \left(\sqrt{x(1-x)} + \frac{1}{n}\right)^r \left[\omega_2^{(r)}\left(\frac{\sqrt{x(1-x)}}{n}\right) + \omega_2^{(r)}\left(\frac{1}{n^2}\right)\right],
 \end{aligned}$$

where M , M_1 , and M_2 are constants. The theorem is completely proved.

Taking into account that always $\omega_2^{(r)}(\delta) \leq 2\omega_1^{(r)}(\delta)$, where $\omega_1^{(r)}(\delta)$ is the modulus of continuity of the derivative $f^{(r)}(x)$, we, replacing in formula (2) $\omega_2^{(r)}(\delta)$ by $2\omega_1^{(r)}(\delta)$, obtain from the theorem proved, as a corollary, Timan' s theorem ⁽³⁾. If, however, we set $\omega_2^{(r)}(\delta) = O(\delta)$, we obtain Theorem 1 from the author' s paper ⁽⁵⁾.

Remark. For purposes of approximation of functions we have also investigated some other polynomial kernels obtained from even trigonometric kernels by the substitution $\cos t = 1 - \frac{1}{2}x^2$.

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Note: Figure translations are in progress. See original paper for figures.

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