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Abstract

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ON THE UNIQUENESS OF DETERMINING THE SHAPE OF AN ATTRACTING BODY FROM THE VALUES OF ITS EXTERNAL POTENTIAL

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P. S. Novikov ([1]) proved the uniqueness of the solution of the inverse problem of potential theory in the class of bodies star-shaped with respect to one and the same point and with the same constant density. L. N. Sretenskii ([2]), using P. S. Novikov' s method (simplified at one point), proved an analogous theorem for bodies possessing parallel middle planes, provided that the center of gravity of each of these bodies lies inside its own body. Yu. A. Shashkin ([3]) established uniqueness theorems in the case of the logarithmic potential for bodies with arbitrary positive mass density.

In the present note, using the method of P. S. Novikov–L. N. Sretenskii, we shall prove the uniqueness of attracting bodies with one and the same constant density under more general assumptions concerning the shape of the bodies than those made in ([1,2]). At the same time we shall show that, in generalized formulations of both theorems, L. N. Sretenskii' s theorem is a limiting case of P. S. Novikov' s theorem.

Let the bodies T_1 and T_2, with the same constant density, occupy bounded open sets in three-dimensional space with surfaces S_1 and S_2, respectively. By S^i_α $(\alpha = 1, 2)$ we denote the part of the surface S_α internal to $\overline{T_1 \cup T_2}$, and by S^i the sum of the sets $S^i_1 \cup S^i_2$. Let S^e be the contour of the body $\overline{T_1 \cup T_2}$, and let S^e_α be the intersection of the surfaces $S^e \cap S_\alpha$. We shall use the usual notation for the closure of a given aggregate:

$$\overline{T_\alpha} = T_\alpha \cup S_\alpha, \qquad \overline{T_1 \cup T_2} = T_1 \cup T_2 \cup S^e.$$

Suppose that the surfaces S_α are piecewise smooth, so that for the domains T_α the formulas for transforming volume integrals into surface integrals hold. Suppose also that each of the surfaces S^e_α has positive measure.

Theorem 1. *Let there exist a point O such that, if R is the radius vector with origin at O, (R, ν) is the scalar product of the vector R with the unit vector*

ν of the exterior normal to S_α, and dS is the surface element of S_α, then the inequality[1]

$$\int_{S^i} |(R,\nu)|\,dS \leq \int_{S^e} |(R,\nu)|\,dS. \tag{1}$$

holds. Then, if the bodies T_α generate identical potentials outside $\overline{T_1}$ and $\overline{T_2}$, they coincide.

Theorem 2. Let the z-axis (with unit vector ν_z) of the rectangular coordinate system x, y, z be chosen so that condition $*$

$$\int_{S^i} |(\nu_z,\nu)|\,dS \leqslant \int_{S^e} |(\nu_z,\nu)|\,dS. \tag{2}$$

is satisfied. Then, if the bodies T_α generate identical potentials outside $\overline{T_1 \cup T_2}$, they coincide.

Remark 1. Let C be an arbitrary fixed point of the domain $T_1 \cup T_2$, and let R_c be the length of the segment OC. Dividing both parts of inequality (1) by R_c, we obtain the equivalent inequality

$$\int_{S^i} \left|\left(\frac{\mathbf{R}}{R_c},\nu\right)\right| dS \leqslant \int_{S^e} \left|\left(\frac{\mathbf{R}}{R_c},\nu\right)\right| dS. \tag{3}$$

Letting the point O tend to infinity along a straight line parallel to the z-axis, by passing to the limit in (3) we obtain (2). Thus, Theorem 2 is a limiting case of Theorem 1.

Remark 2. Inequality (1) is satisfied, in particular, if the section $\overline{T_1} \cap \overline{T_2}$ is star-shaped with respect to the point O, i.e., if the surface S^i in a polar coordinate system with pole at O is given by the equation $r = R(\theta,\varphi)$, where R is a single-valued function of the polar angles. In this case $(\mathbf{R},\nu) \geq 0$ on S^i, and (1) is a consequence of the inequality

$$\frac{1}{3}\int_{S^i} (\mathbf{R},\nu)\,dS \leqslant \frac{1}{3}\int_{S^e} (\mathbf{R},\nu)\,dS,$$

expressing the fact that the volume of the cone with vertex at O and base on S^i does not exceed the volume of the body $T_1 \cup T_2$.

Inequality (2) is satisfied, in particular, if every straight line parallel to the z-axis intersects the surface S^i in (no more than) two points, or has a common segment with S^i. In this case inequality (2) expresses the fact that the sum of

[1] If the bodies T_α are externally tangent, then the corresponding integral on the left-hand side of inequalities (1) and (2) must be taken twice–over S^i_1 and over S^i_2.

the areas of the projections of the surface S^i onto the plane xy does not exceed the sum of the areas of the projections of the surface S^e.

It is not difficult to see that the special examples given are far from exhausting all cases in which inequalities (1) and (2) hold.

Remark 3. Theorems 1 and 2 remain valid also in the case when the density of the bodies T_α is a function, respectively, of θ, φ or of x, y.

Remark 4. The assumption that the surfaces S^e_α have positive measures may be replaced by the requirement that equality should not be allowed in inequalities (1) and (2). If both requirements are not fulfilled, Theorem 1 is not valid. This is seen from the following example: the sphere $0 \le r \le 1$ and the spherical shell $1 \le r \le \sqrt[3]{2}$ (with the same constant density) have the same external potential, although they satisfy inequality (1), which in this example becomes the equality $8\pi = 8\pi$.

Proof of Theorem 1. If $U(x, y, z)$ is a harmonic function in $T_1 \cup T_2$ and has continuous partial derivatives in $\overline{T_1 \cup T_2}$, then a simple consequence of Green' s formula is the equality ([1])

$$\iiint_{T_1} U(r,\theta,\varphi)\, r^2\, dr\, \sin\theta\, d\theta\, d\varphi = \iiint_{T_2} U(r,\theta,\varphi)\, r^2\, dr\, \sin\theta\, d\theta\, d\varphi \qquad (4)$$

(the pole of the polar coordinate system is chosen at the point O).

* See the footnote to Theorem 1.

Let the function $H(r, \theta, \varphi)$ be harmonic and have continuous partial derivatives up to and including the second order in $\overline{T_1 \cup T_2}$. Applying formula (4) to the function

$$U(r,\theta,\varphi) = 3H(r,\theta,\varphi) + r\frac{\partial H(r,\theta,\varphi)}{\partial r}.$$

Taking into account that

$$\left(3H + r\frac{\partial H}{\partial r}\right) r^2 = \frac{\partial}{\partial r}(r^3 H)$$

and

$$\operatorname{sign}(R,\nu) \cdot R^3 \sin\theta\, d\theta\, d\varphi = (R,\nu) dS$$

($\operatorname{sign} t = 1$ for $t > 0$; $\operatorname{sign} t = 0$ for $t = 0$; $\operatorname{sign} t = -1$ for $t < 0$), we obtain

$$\int_{S_1} H(R)(R,\nu)\,dS = \int_{S_2} H(R)(R,\nu)\,dS. \tag{5}$$

Representing $\overline{T_1 \cup T_2}$ as the intersection of a sequence of open sets and solving the Dirichlet problem for these sets, one can extend equality (5) to all functions H that are piecewise continuous in $T_1 \cup T_2$ (and harmonic in $T_1 \cup T_2$).

We shall show that the solution $H(R)$ of the Dirichlet problem for the aggregate $T_1 \cup T_2$ with boundary conditions on S^e *

$$H(R) = \begin{cases} -\operatorname{sign}(R,\nu), & \text{for } R \in S_1^e, \\ \operatorname{sign}(R,\nu), & \text{for } R \in S_2^e \end{cases} \tag{6}$$

does not satisfy (5), if the bodies T_1 and T_2 do not coincide.

Indeed, for the function $H(R)$ given by (6), equality (5) takes the form

$$\int_{S_1^i} H(R)(R,\nu)\,dS - \int_{S_2^i} H(R)(R,\nu)\,dS = \int_{S^e} |(R,\nu)|\,dS.$$

On the other hand, on the basis of the maximum principle for the modulus, the left-hand side of the last equality can be estimated as follows:

$$\int_{S_1^i} H(R)(R,\nu)\,dS - \int_{S_2^i} H(R)(R,\nu)\,dS < \int_{S^i} |(R,\nu)|\,dS,$$

which, together with inequality (1), leads to a contradiction. Theorem 1 is proved.

The proof of Theorem 2 is carried out analogously. Instead of the function

$$r\frac{\partial H}{\partial r} + 3H$$

one uses the function

$$\frac{\partial H}{\partial z}.$$

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CITED LITERATURE

[1] P. S. Novikov, *DAN*, 18, No. 3 (1938).
[2] L. N. Sretenskii, *DAN*, 99, No. 1 (1954).
[3] Yu. A. Shashkin, *DAN*, 115, No. 1 (1957).

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* Without loss of generality one may assume that the intersection of the surfaces $S_1^e \cap S_2^e$ has measure zero. Otherwise the theorem is trivial.

Note: Figure translations are in progress. See original paper for figures.