MATHEMATICS

A. F. ZOLIN

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. F. ZOLIN

ON THE APPROXIMATE SOLUTION OF THE POLYHARMONIC PROBLEM

(Presented by Academician S. L. Sobolev on 5 VI 1958)

In the present note the principal boundary-value problem for the equation

$$\Delta^p U = \sum_{\substack{\alpha,\beta=0,p \\ \alpha+\beta=p}} \frac{p!}{\alpha!\beta!} \frac{\partial^{2p} U}{\partial x^{2\alpha} \partial y^{2\beta}} = 0. \tag{1}$$

is solved approximately.

The function U, therefore, must, in addition to (1), satisfy the conditions

$$U\Big|_\Gamma = f_0(s), \quad \frac{\partial U}{\partial \nu}\Big|_\Gamma = f_1(s), \dots, \quad \frac{\partial^{p-1} U}{\partial \nu^{p-1}}\Big|_\Gamma = f_{p-1}(s), \tag{2}$$

where $f_0, f_1, \dots, f_{p-1}$ are prescribed continuous functions on the boundary Γ of the given simply connected domain Ω; ν is the direction of the outward normal to Γ.

This problem was studied from the point of view of existence, uniqueness, and possible methods of its solution by S. L. Sobolev (under very general assumptions concerning the domain and its boundary) ([1]), and also by I. N. Vekua ([2]), N. Meiman ([3]), and others.

We shall assume that the boundary Γ consists of a finite number of piecewise-smooth curves.

In what follows it will be more convenient for us to deal with the following boundary conditions

for p even   for p odd

$$
\begin{array}{ll}
U\big|_\Gamma = \varphi_0(s), & U\big|_\Gamma = \varphi_0(s), \\
\left.\dfrac{\partial U}{\partial \nu}\right|_\Gamma = \varphi_1(s), & \left.\dfrac{\partial U}{\partial \nu}\right|_\Gamma = \varphi_1(s), \\
\Delta U\big|_\Gamma = \varphi_2(s), & \Delta U\big|_\Gamma = \varphi_2(s), \\
\left.\dfrac{\partial \Delta U}{\partial \nu}\right|_\Gamma = \varphi_3(s), & \left.\dfrac{\partial \Delta U}{\partial \nu}\right|_\Gamma = \varphi_3(s), \\
\cdots\cdots\cdots & \cdots\cdots\cdots \\
\Delta^{\left[\frac{p-1}{2}\right]} U\big|_\Gamma = \varphi_{p-2}(s), & \cdots\cdots\cdots \\
\left.\dfrac{\partial \Delta^{\left[\frac{p-1}{2}\right]} U}{\partial \nu}\right|_\Gamma = \varphi_{p-1}(s), & \Delta^{\frac{p-1}{2}} U\big|_\Gamma = \varphi_{p-1}(s).
\end{array}
\tag{2'}
$$

Here $\varphi_0, \varphi_1, \dots, \varphi_{p-1}$ are certain continuous functions that can be expressed in terms of the functions $f_0, f_1, \dots, f_{p-1}$, and the square brackets denote the greatest integer part of a number. It is easy to verify that conditions (2) and (2′) are equivalent in the sense that a function U satisfying one set of conditions satisfies the other, and conversely.

Following the general idea of the least-squares method [4], in this note approximate solutions of the problem are constructed in such a way that they satisfy equation (1) exactly and satisfy the conditions (2′) approximately, i.e., a method is indicated for constructing a sequence of polyharmonic functions converging in Ω to the exact solution of the problem under consideration.

Naturally, as a system of approximating functions one should take functions that are simplest in form. Therefore, as such functions we shall take polyharmonic polynomials, which we write, with the aid of polar coordinates, in the form

$$
u_{pn} = \sum_{\substack{i=1,p \\ j=0,n}} \rho^{2(i-1)+j} \left(a_{ij} \cos j\theta + b_{ij} \sin j\theta \right). \tag{3}
$$

Let us consider the set of all regular [2] polyharmonic functions defined in Ω, and turn it into a Hilbert space by defining the scalar product as follows:

$$
(U, V) = \sum_{k=0}^{\left[\frac{p-1}{2}\right]}{}' \int_\Gamma \left(\Delta^k U \Delta^k V + \frac{\partial \Delta^k U}{\partial \nu} \frac{\partial \Delta^k V}{\partial \nu} \right) ds, \tag{4}
$$

where the prime means that, in the case of odd p, the last term is absent. It is not difficult to verify that expression (4) satisfies the axioms of a scalar product. The metric in which we shall carry out the approximation will be defined by means of the norm, which therefore takes the form

$$\|U\|^2 = \sum_{k=0}^{\left[\frac{p-1}{2}\right]}{}' \int_\Gamma \left[\left(\Delta^k U\right)^2 + \left(\frac{\partial \Delta^k U}{\partial \nu}\right)^2 \right] ds. \tag{5}$$

We first note the following fact.

Lemma. *The polyharmonic polynomials (3), considered in $\overline{\Omega}$, are everywhere dense in the space E together with their partial and mixed derivatives of arbitrary order.*

This proposition can be proved by using the Chebyshev metric. It will then obviously follow that it is also true in the metric defined by the norm (5).

The proof of the lemma follows from the representation of an arbitrary polyharmonic function in Almansi' s form

$$U = \sum_{i=0}^{p-1} \rho^{2i} U_i,$$

where the U_i are certain harmonic functions, and from the fact that any regular harmonic function can be approximated uniformly without bound by harmonic polynomials, and its derivatives by the corresponding derivatives of the same harmonic polynomials. The latter circumstance is a consequence of Walsh' s generalized theorem, which establishes the fundamental possibility of simultaneous approximation, by polynomials in a complex variable and by their corresponding derivatives, of functions of a complex variable analytic in Ω, continuous in $\overline{\Omega}$, and of their derivatives.

Corollary. From the lemma there follows the closedness of the polynomials (3), together with their expressions (2′), in the mean on Γ in the space of functions continuous on Γ, as well as their closedness in $\overline{\Omega}$ in the sense of our metric and in E.

As an approximate solution of the problem we take a polynomial U_{pn} of the form (3) which minimizes $\|U - u_{pn}\|^2$. We note that this minimum is in principle easily realized. By the lemma and its corollary, the polynomial U_{pn} converges in the mean, together with its expressions (2′), on the boundary Γ to the corresponding boundary functions.

Theorem. At any interior point of the domain Ω, the polynomials U_{pn} converge in the ordinary sense, in any closed domain $\overline{D}$ lying inside Ω, uniformly to the exact solution of the problem under consideration as n increases without bound.

Apply to the difference $U - U_{pn}$ the formula expressing the value of a polyharmonic function inside a domain in terms of its boundary values (2′) and the polyharmonic Green function ([2,5])

$$U - U_{pn} = \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \int_\Gamma \left[\Delta^k (U - U_{pn}) \frac{\partial \Delta^{p-k-1} G}{\partial \nu} - \frac{\partial \Delta^k (U - U_{pn})}{\partial \nu} \Delta^{p-k-1} G \right] ds.$$

It follows that

$$|U(x,y) - U_{pn}(x,y)| \leq$$

$$\leq \sum_{k=0}^{\left[\frac{p-1}{2}\right]} \left\{ \left[\int_\Gamma (\Delta^k U - \Delta^k U_{pn})^2 \, ds \right]^{1/2} \left[\int_\Gamma \left(\frac{\partial \Delta^{p-k-1} G}{\partial \nu} \right)^2 ds \right]^{1/2} + \right.$$

$$\left. + \left[\int_\Gamma \left(\frac{\partial \Delta^k U}{\partial \nu} - \frac{\partial \Delta^k U_{pn}}{\partial \nu} \right)^2 ds \right]^{1/2} \left[\int_\Gamma (\Delta^{p-k-1} G)^2 \, ds \right]^{1/2} \right\}. \qquad (6)$$

The functions

$$\Phi_{1k}(x,y) = \left[\int_\Gamma \left(\frac{\partial \Delta^{p-k-1} G}{\partial \nu} \right)^2 ds \right]^{1/2}, \qquad \Phi_{2k}(x,y) = \left[\int_\Gamma (\Delta^{p-k-1} G)^2 \, ds \right]^{1/2}$$

are continuous and, consequently, bounded in $\overline{D}$. The constant coefficients multiplying these functions in (6) tend to zero as n increases, which proves the theorem.

Institute of Mathematics and Mechanics
named after V. I. Romanovskii
Academy of Sciences of the UzSSR

Received
17 V 1958

REFERENCES

[1] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950.
[2] I. N. Vekua, *New Methods for Solving Elliptic Equations*, Moscow-Leningrad, 1948.
[3] N. Meiman, DAN, **33**, No. 4, 275 (1941).
[4] S. G. Mikhlin, *Direct Methods in Mathematical Physics*, Moscow-Leningrad,

1950.
[5] M. Nicolescu, *Les fonctions polyharmoniques*, Paris, 1936.

Note: Figure translations are in progress. See original paper for figures.