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Abstract

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PHYSICS

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ON A LINEAR BOUNDARY-VALUE PROBLEM OF GENERALIZED HYDRODYNAMICS

Toward the Theory of the Ultrasonic Interferometer

(Presented by Academician N. N. Bogolyubov on 22 VII 1958)

1. The theory of the ultrasonic interferometer consists of two parts. The first part is the solution of the problem of the propagation of small perturbations in a medium and of the corresponding boundary conditions; the second part is the calculation of the electrical circuit of the interferometer. These problems for the case of gases were solved in papers ([1,2]), and the solution of the first problem was based on the usual equations of acoustics. Since the equations of hydrodynamics in the case of gases are a consequence of an approximate solution of the Boltzmann kinetic equation ([3]), the theories considered in ([1,2]) are limited by the requirement
$\varepsilon = \nu\dfrac{\mu}{p} \ll 1$*. On the other hand, experimental measurements of the translational dispersion of ultrasound in monatomic gases were carried out in the following ranges of ε: in paper ([4]), $\varepsilon \lesssim 0.16$ (or $r \gg 1$, $r = 1/2\pi\varepsilon$); in ([5]), $\varepsilon \lesssim 1.6$ ($r \gg 0.1$); in ([6]), $\varepsilon \lesssim 16$ ($r \gg 0.01$). Consequently, if Bojer reached the limit of applicability of ordinary hydrodynamics (including the Burnett approximation), then in papers ([5,6]) the authors went beyond the region in which hydrodynamic approximations according to Chapman can be applied. Thus the question arises of a theory of the ultrasonic interferometer in the case $\varepsilon \gg 1$.

In ([7,8]) we considered the apparatus of generalized hydrodynamics, free from restrictions on the magnitude of ε, as well as the question of boundary conditions. The aim of the present work is to solve the simplest boundary-value problem of generalized hydrodynamics, which constitutes the acoustic part of the theory of the ultrasonic interferometer.

2. Consider the following scheme of an interferometer: there are two infinite parallel planes, reflecting the gas atoms specularly and not conducting heat; between them there is a monatomic gas; the plane $x = 0$ (the emitter) oscillates according to a harmonic law with infinitely small amplitude, while the plane

$x = a$ is motionless. In this case the behavior of the gas is governed by the linear one-dimensional equations of generalized hydrodynamics (see ([7])). For simplicity we shall consider these equations with accuracy up to moments of the second order, i.e., we shall neglect the effects of heat conduction in the gas. Then the initial equations will be:

$$\frac{\partial n'}{\partial t} + (c_e)_0 \frac{\partial u'}{\partial x} = 0, \qquad \frac{\partial u'}{\partial t} + (c_e)_0 \frac{\partial P'_{11}}{\partial x} = C, \tag{1}$$

$$\frac{\partial P'}{\partial t} + \frac{5}{3}(c_e)_0 \frac{\partial u'}{\partial x} = 0, \qquad \frac{\partial P'_{11}}{\partial t} + 3(c_e)_0 \frac{\partial u'}{\partial x} = \frac{6}{\tau}\left(P' - P'_{11}\right),$$

* ν is the prescribed frequency of the sound wave; p/μ is the "proper" frequency of the gas, i.e., the frequency of collisions of gas particles with one another; μ is the coefficient of viscosity; p is the pressure of the gas.

where $n' = \Delta n/n_0$; $u' = \Delta u/(c_e)_0$; $P'_{11} = \Delta P_{11}/P_0$; $P' = \Delta P/P_0$; $(c_e)_0 = \sqrt{kT_0/m}$ (here the quantities with subscript zero refer to the stationary state, while quantities of the type Δn are deviations from the stationary state); n is the number density of particles; u is the mean velocity of the gas particles; P_{11} is a component of the stress tensor; $P = \frac{1}{3}\sum_i P_{ii}$; τ is a time of the order of the relaxation time (see (7)).

Since in this problem we do not consider phenomena of heat-energy transfer, the boundary conditions will be (see (8)):

$$u'\big|_{x=0} = \frac{1}{(c_e)_0}\,\mathrm{Re}\,\dot{\xi}, \qquad u'\big|_{x=a} = 0, \tag{2}$$

where $\dot{\xi} = \dot{\xi}_0 e^{i\omega t}$; $\dot{\xi}_0$ is the amplitude of the emitter velocity (a real quantity); ω is the angular frequency (a real quantity).

In accordance with (2), we shall seek the solution of system (1) in the form

$$n' = \mathrm{Re}\left[e^{i\omega t} n'(x)\right], \qquad u' = \mathrm{Re}\left[e^{i\omega t} u'(x)\right],$$

$$P' = \mathrm{Re}\left[e^{i\omega t} P'(x)\right], \qquad P'_{11} = \mathrm{Re}\left[e^{i\omega t} P'_{11}(x)\right]. \tag{3}$$

Substituting (3) into (1), we obtain:

$$i\omega n'(x) + (c_e)_0 \frac{du'(x)}{dx} = 0, \qquad i\omega u'(x) + (c_e)_0 \frac{dP'_{11}(x)}{dx} = 0,$$

$$i\omega P'(x) + \frac{5}{3}(c_e)_0 \frac{du'(x)}{dx} = 0, \tag{4}$$

$$i\omega P'_{11}(x) + 3(c_e)_0 \frac{du'(x)}{dx} = \frac{6}{\tau}\left[P'(x) - P'_{11}(x)\right].$$

The first and third, and the third and fourth equations of (4) give, respectively:

$$P'(x) = \frac{5}{3}n'(x), \qquad P'_{11} = \frac{1}{K^2}P'(x), \tag{5}$$

where

$$K^2 = \frac{1 + i\frac{1}{r}}{1 + i\frac{9}{5r}}, \qquad r = \frac{1}{2\pi\varepsilon}.$$

The third equation of (4) and the second equations of (4) and (5) give

$$\frac{du'(x)}{dx} + a_{11}u'(x) + a_{12}P'(x) = 0,$$

$$\frac{dP'(x)}{dx} + a_{21}u'(x) + a_{22}P'(x) = 0, \tag{6}$$

where $a_{11} = a_{22} = 0$; $a_{12} = i\sqrt{3/5}\,\beta_0$; $a_{21} = i\sqrt{5/3}\,\beta_0 K^2$; $\beta_0 = \omega/V_{\text{ad}}$, $V_{\text{ad}} = \sqrt{5/3}\,(c_e)_0$.

The characteristic equation of system (6) is

$$(ik)^2 - a_{12}a_{21} = 0. \tag{7}$$

The solutions of (7) are

$$k_1 = +\beta_0 K, \qquad k_2 = -\beta_0 K, \qquad K = +\sqrt{K^2}. \tag{8}$$

Since the roots (8) of the characteristic equation are distinct, the general solution of (6) is the system

$$u'(x) = C^{(1)}\left(-\frac{1}{\sqrt{5/3}\,K}\right)e^{ik_1x} + C^{(2)}\left(+\frac{1}{\sqrt{5/3}\,K}\right)e^{ik_2x},$$

$$P'(x) = C^{(1)}e^{ik_1x} + C^{(2)}e^{ik_2x}. \tag{9}$$

The constants $C^{(1)}$ and $C^{(2)}$ in (9) are determined from the boundary conditions (2):

$$C^{(1)} = \sqrt{5/3}\, K \left(\frac{\dot{\xi}_0}{(c_e)_0} \right) \frac{e^{-ik_1 a}}{e^{ik_1 a} - e^{-ik_1 a}},$$

$$C^{(2)} = \sqrt{5/3}\, K \left(\frac{\dot{\xi}_0}{(c_e)_0} \right) \frac{e^{ik_1 a}}{e^{ik_1 a} - e^{-ik_1 a}}. \tag{10}$$

Finally, from (3), (5), (9), (10) we obtain the solution of the boundary-value problem posed:

$$\Delta\rho = \Delta p \, \frac{1}{V_{\text{ad}}^2}, \qquad \Delta u = \dot{\xi}_0 \operatorname{Re} \left\{ e^{i\omega t} \frac{e^{-ik_1 x} - e^{-ik_1 (2a-x)}}{1 - e^{-ik_1 2a}} \right\},$$

$$\Delta p = \rho_0 V_{\text{ad}} \dot{\xi}_0 \operatorname{Re} \left\{ e^{i\omega t} K \frac{e^{-ik_1 x} + e^{-ik_1 (2a-x)}}{1 - e^{-ik_1 2a}} \right\}, \tag{11}$$

$$\Delta P_{11} = \rho_0 V_{\text{ad}} \dot{\xi}_0 \operatorname{Re} \left\{ e^{i\omega t} \frac{1}{K} \frac{e^{-ik_1 x} + e^{-ik_1 (2a-x)}}{1 - e^{-ik_1 2a}} \right\}.$$

Let us note that the usual consequence of adiabaticity is also preserved in our case: $\Delta P / \Delta\rho = V_{\text{ad}}^2$. However, this expression will give the phase velocity of sound only for $\varepsilon \to 0$ ($r \to \infty$) (see (8, 5)). We also emphasize that, in contrast to ordinary acoustics, generally speaking, $\Delta P_{11} \neq \Delta P$. Indeed, from the definition $3\Delta P = \sum_i \Delta P_{ii}$ (see (7)), the condition of one-dimensionality and (5), we obtain: for $r \to \infty$, $\Delta P_{11} = \Delta P_{22} = \Delta P_{33} = \Delta P$; for $r \to 0$, $\Delta P_{11} = \frac{9}{5}\Delta P$, $\Delta P_{22} = \Delta P_{33} = \frac{3}{5}\Delta P$. Thus, the expression for ΔP_{11} (and not ΔP!) gives the excess (relative to the stationary value) pressure in the direction perpendicular to the boundary planes.

In order to make direct use of the results of paper (1), let us represent ΔP_{11} from (11) in Hubbard' s form. Introduce the usual notation:

$$k_1 = +(\beta - i\alpha), \qquad K = + \left(\frac{\beta}{\beta_0} - i \frac{\alpha}{\beta_0} \right), \qquad \beta = \frac{\omega}{V}, \qquad \beta_0 = \frac{\omega}{V_{\text{ad}}}. \tag{12}$$

Then, taking into account (12) and the fact that $\dot{\xi} = i\omega\xi$, $\omega\xi = -i\dot{\xi}$, from (11) we obtain:

$$\Delta P_{11} = \rho_0 V_{\text{ad}} \operatorname{Re}\{\mathcal{F}\dot{\xi} + Q\omega\xi\}, \tag{13}$$

where

$$\mathcal{F} = \left(\frac{\beta_0}{\beta}\right) \frac{P_{\text{H}} + Q_{\text{H}}(\alpha/\beta)}{1 + (\alpha/\beta)^2}, \qquad Q = \left(\frac{\beta_0}{\beta}\right) \frac{Q_{\text{H}} - P_{\text{H}}(\alpha/\beta)}{1 + (\alpha/\beta)^2},$$

$$P_{\text{H}} = \frac{[e^{-\alpha x} - e^{-(4a-x)\alpha}]\cos(\beta x) + [e^{-\alpha(2a-x)} - e^{-\alpha(2a+x)}]\cos[\beta(2a-x)]}{1 - 2e^{-2a\alpha}\cos(2a\beta) + e^{-4a\alpha}},$$

$$Q_{\text{H}} = \frac{[e^{-\alpha x} + e^{-(4a-x)\alpha}]\sin(\beta x) + [e^{-\alpha(2a-x)} + e^{-\alpha(2a+x)}]\sin[\beta(2a-x)]}{1 - 2e^{-2a\alpha}\cos(2a\beta) + e^{-4a\alpha}};$$

the values of β_0/β and α/β, in view of the unwieldiness of the explicit formulas (see (5)), are given in Table 1.

Table 1

r	$\left(\frac{\beta_0}{\beta}\right) = \frac{V}{V_{\text{ad}}}$	$\left(\frac{\alpha}{\beta}\right)$	r	$\left(\frac{\beta_0}{\beta}\right) = \frac{V}{V_{\text{ad}}}$	$\left(\frac{\alpha}{\beta}\right)$
0.01	1.3416	0.002	0.5	1.2987	0.096
0.02	1.3416	0.004	1.0	1.2184	0.140
0.05	1.3412	0.011	2.0	1.1070	0.135
0.1	1.3397	0.021	5.0	1.0237	0.074
0.2	1.3339	0.044	10.0	1.0063	0.039

3. From consideration of (13) it follows: 1) in the case $\varepsilon \to 0$ ($r \to \infty$), $\mathcal{P}^0 \to P_n$ and $Q \to Q_n$, i.e., in the classical region (13) coincides with formula (11) from (1); 2) since (13) coincides in form with (11) from (1) for all values of r, the further calculation of the velocity and attenuation of sound from the readings of the electrical indicator can be carried out according to Hubbard' s scheme; 3) the principal expected result is a decrease in the attenuation of ultrasound when $1 > r \to 0$ (or $0.16 < \varepsilon \to \infty$).

Let us emphasize that the results of the present work are based on the Boltzmann kinetic equation for monatomic gases.

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Note: Figure translations are in progress. See original paper for figures.