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Abstract

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SYSTEMS OF COMPLETE SINGULAR INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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In papers ([1]), on the basis of the theory of the Fourier transform and the theory of singular integral equations with the Cauchy kernel, singular integral equations of convolution type with one unknown function were studied. In the present note, systems of such equations are considered. We write them in abbreviated form as

$$f(x) + \frac{1}{\sqrt{2\pi}} \int_0^\infty k_1(x-t) f(t)\, dt +$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 k_2(x-t) f(t)\, dt + Tf = g(x), \qquad -\infty < x < \infty; \tag{A}$$

$$f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_1(x-t) f(t)\, dt + Tf = g(x), \qquad x > 0;$$

$$f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_2(x-t) f(t)\, dt + Tf = g(x), \qquad x < 0, \tag{B}$$

where $k_\alpha(x) = \|k_{ij}^\alpha(x)\|$ $(-\infty < x < \infty,\ \alpha = 1,2)$ are square matrix functions with elements from $L_1(-\infty,\infty)$; $g(x) = \{g_i(x)\}$; $f(x) = \{f_i(x)\}$ are n-dimensional vector functions with elements from $L_2(-\infty,\infty)$; $T = \|T_{ij}\|$ is a matrix whose elements are completely continuous operators. We assume that

$$\det[E + K_\alpha(x)] \neq 0 \qquad \left(K_\alpha(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_\alpha(t) e^{ixt}\, dt\right).$$

A special case of systems (A) and (B)—the so-called systems of characteristic equations ($T \equiv 0$)—was considered by us in note ([2]).

As the apparatus for the investigation of the indicated systems of equations, we use the Fourier transform ([3]), matrix theory, and the theory of boundary value problems for analytic functions for a system of unknown functions ([4], [5]).

§ 1. Denoting

$$a(x) = \frac{1}{2}[k_1(x) + k_2(x)], \qquad b(x) = \frac{1}{2}[k_1(x) - k_2(x)], \tag{1}$$

we write system (A) in the form

$$lf \equiv f(x)+\frac{1}{\sqrt{2\pi}}\int_\Gamma a(x-t)f(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_\Gamma b(x-t)f(t)\,\mathrm{sgn}\,t\,dt+Tf = g(x), \tag{2}$$

where the contour Γ is the real axis. Define the operator l^*, adjoint to l, using the usual definition of the adjoint operator from the scalar product:

$$l^*\varphi \equiv \varphi(x) + \frac{1}{\sqrt{2\pi}}\int\limits_\Gamma a^*(t-x)\varphi(t)\,dt + \frac{\mathrm{sgn}\,x}{\sqrt{2\pi}}\int\limits_\Gamma b^*(t-x)\varphi(t)\,dt + T^*\varphi, \tag{3}$$

where a^*, b^*, T^* are matrices Hermitian adjoint, respectively, to a, b, T.

Taking (1) into account, we write the operator (3) in the form of a "paired" vector operator

$$l^*\varphi \equiv \begin{cases} \varphi(x) + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int\limits_\Gamma k_1^*(t-x)\varphi(t)\,dt + T^*\varphi, & x > 0; \\ \varphi(x) + \dfrac{1}{\sqrt{2\pi}}\displaystyle\int\limits_\Gamma k_2^*(t-x)\varphi(t)\,dt + T^*\varphi, & x < 0. \end{cases}$$

It follows from this that the equation adjoint to equation (A) is an equation of type (B). Similarly, one can show that the equation adjoint to equation (B) is an equation of type (A).

Definitions. By the **total index** of the operator l, or of the equation $lf = g$, we shall mean the integer

$$\varkappa = \frac{1}{2\pi i}\left[\ln\frac{\det[E + K_2(x)]}{\det[E + K_1(x)]}\right]_\Gamma,$$

where $[\,]_\Gamma$ denotes the increment of the function enclosed in brackets upon a complete traversal of the contour Γ.

By the **partial indices** of the operator l we shall mean the partial indices of the Riemann boundary-value problem corresponding to the characteristic singular equation $l^0 f = g$ ([2]).

The following Noether theorems hold:

Theorem 1. The homogeneous vector equation $lf = 0$ has only a finite number of linearly independent solutions.

Theorem 2. In order that the equation $lf = g$ have a solution, it is necessary and sufficient that $(g, \psi_i) = 0$, where ψ_i is a complete system of solutions of the adjoint homogeneous equation $l^*\psi = 0$.

Theorem 3. The difference between the numbers of solutions of the homogeneous equation $lf = 0$ and of the homogeneous adjoint equation $l^*\psi = 0$ depends only on the characteristic part l^0 of the operator l and is equal to the total index of the operator l.

§ 2. We shall find an operator p such that $plf = pg$ is a Fredholm equation. In this case we shall call p a regularizer, or a regularizing operator, and with respect to the operator l and the equation $lf = g$ we shall say that they admit regularization.

For the singular equation (2) we shall seek the regularizing operator in the form

$$p\varphi \equiv \varphi(x) + \frac{1}{\sqrt{2\pi}} \int_{\Gamma} \tilde{a}(x-t)\varphi(t)\,dt + \frac{1}{\sqrt{2\pi}} \int_{\Gamma} \tilde{b}(x-t)\varphi(t)\,\mathrm{sgn}\,t\,dt, \qquad (4)$$

where

$$\tilde{a}(x) = \frac{1}{2}\,[r_1(x) + r_2(x)], \qquad \tilde{b}(x) = \frac{1}{2}\,[r_1(x) - r_2(x)] \qquad (5)$$

are certain matrices to be determined. Forming the composition plf of the operators (4) and (2), we obtain:

In order that $plf = pg$ be a Fredholm equation, it is necessary and sufficient that $r_1(x), r_2(x)$ be solutions of the integral matrix equations

$$r_\alpha(x) + k_\alpha(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_\alpha(x-t) r_\alpha(t)\,dt = 0, \qquad \alpha = 1, 2. \qquad (6)$$

Definition. The matrices r_1, r_2 satisfying the last condition will be called the **resolvents** of the corresponding kernels k_1, k_2.

§ 3. Consider the question of the possibility of constructing, for our equation $lf = g$, such a regularizing operator p that the Fredholm equation $plf = pg$ is equivalent to the original equation $lf = g$. In this case p will be called an equivalently regularizing operator for l, or an **equivalent regularizer**.

We first make two remarks (see, for example, (6)).

I. If $p\varphi = 0$ is an unsolvable equation, then the Fredholm equation $plf = pg$ is equivalent to the original equation $lf = g$.

Thus, if there exists a regularizer having no eigenfunctions, then this operator will be an equivalent regularizer.

II. If l^* is the operator adjoint to l, and the equation $lf = g$ is solvable, then it is equivalent to the equation $l^*lf = l^*g$. If, moreover, l is a real operator, then l^* will also be a regularizer.

We now proceed to the construction of an equivalent regularizer.

Theorem. *If the system of singular integral equations is solvable, then it admits an equivalently regularizing operator.*

Proof. As was shown above, a regularizing operator p for equation (2) may be taken in the form (4). Denoting the partial indices of the operator p by $\nu_1, \nu_2, \dots, \nu_n$, we form a new operator q so that its partial indices are $\nu_1 - m, \nu_2 - m, \dots, \nu_n - m$. Such an operator is

$$q\varphi \equiv p\varphi - \frac{1}{2}\sum_{k=1}^{m} \frac{2^k C_m^k}{(k-1)!} \int_x^{\infty} (x-t)^{k-1} e^{x-t} \Bigg[\varphi(t)(1-\operatorname{sgn} t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_2(t-\tau)\varphi(\tau)(1-\operatorname{sgn}\tau)\, d\tau\Bigg],$$

where C_m^k are binomial coefficients. This may be verified by reducing the homogeneous equation $q\varphi = 0$ to a Riemann problem by the method indicated in (2).

Choose the number m so that all the expressions $\nu_1 - m, \dots, \nu_n - m$ are non-positive. Then the homogeneous equation $q\varphi = 0$ is solvable, and, by virtue of remark I, the equation $qlf = qg$ is equivalent to the equation $lf = g$. Using the properties of the resolvents (6), we obtain explicitly the equation

$$qlf \equiv f(x) - \frac{1}{2}\sum_{k=1}^{m} \frac{2^k C_m^k}{(k-1)!} \int_x^{\infty} (x-t)^{k-1} e^{x-t} f(t)(1-\operatorname{sgn} t)\, dt + T_2 f = qg, \quad (7)$$

where T_2 is a completely continuous operator. By virtue of remark II, the latter is equivalent to the equation

$$(ql)^*(ql)f = (ql)^* qg, \quad (8)$$

which is Fredholm, since $(ql)^0$ is an essential operator.

By formula (3) we define the operator $(ql)^*$:

$$(ql)^*\psi \equiv \psi(x) + \frac{1}{2}\sum_{k=1}^{m} \frac{(-2)^k C_m^k}{(k-1)!} \int_{-\infty}^{x} (x-t)^{k-1} e^{-(x-t)} \psi(t)(1-\operatorname{sgn} x)\, dt + T_2^*\psi.$$

Now, forming the composition of the operators (8), we obtain the Fredholm equation

$$(ql)^*(ql)f \equiv f(x) + T_3 f = (ql)^* qg(x). \tag{8'}$$

Since equation (7), by virtue of Remark I, is equivalent to the original $lf = g$, it follows that, if the original system of equations is solvable, it has the equivalent regularizer $(ql)^*q$ and is equivalent to the system of Fredholm integral equations $(8')$.

Thus, a proof has been given of the existence of an equivalent regularizer for every solvable system of singular equations of convolution type (A), and at the same time its explicit expression has been constructed.

The system of singular equations (B) is investigated analogously.

In conclusion I express my deep gratitude to my supervisor, Prof. F. D. Gakhov.

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Note: Figure translations are in progress. See original paper for figures.