ON THE LEVITZKI PROBLEM

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Abstract

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MATHEMATICS

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An associative ring S is called a **nil-ring** if every element of the ring S is nilpotent. Levitzki (^{4}) posed the problem: is every nil-ring nilpotent? This problem was solved affirmatively by Levitzki himself (^{5}) for the case when the indices of nilpotency of all elements of the ring S are bounded in the aggregate. Later Kaplansky (^{2}), dealing with the solution of a more general problem of Kurosch (^{3}), extended Levitzki' s result to nil-rings with polynomial identities. In the present note an affirmative solution of Levitzki' s problem is given for a broader class of rings introduced for consideration by Dresin (^{1}).

Let $\Lambda = \{\lambda_i\}$, $i = 1, 2, \dots, h$, be some set of variables, and let $\pi(\lambda) = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$ be some monomial in these variables. Denote by $T_\pi(\lambda)$ the set of all monomials in the set Λ of degree $\geq k$ and distinct from $\pi(\lambda)$.

For any sequence of elements $\{x_i\}$, $i = 1, 2, \dots, h$, of the ring S, by $\pi(x)$ we denote the element $x_{i_1}x_{i_2}\cdots x_{i_k}$, and by $T_\pi(x)$ the set of elements of the ring S obtained by replacing all variables λ_i by the corresponding elements x_i in each monomial of the set $T_\pi(\lambda)$.

If there exists a monomial $\pi(\lambda)$ such that, for any choice of elements x_i, $i = 1, 2, \dots, h$, of the ring S, the element $\pi(x)$ lies in the right ideal generated by the set $T_\pi(x)$, then the monomial $\pi(\lambda)$ is called a **strictly supporting monomial** of the ring S, and the ring S a **ring with a strictly supporting monomial**. For brevity we shall call such rings SP-rings.

Dresin (^{1}) showed that the class of SP-rings contains rings with the minimal condition for right ideals and rings with polynomial identities. In the same work it is shown that, for any SP-ring, the monomial $\pi(\lambda)$ may be assumed linear in each of the variables λ_i. Under certain strong restrictions Dresin, making use, in essence, of Kaplansky' s methods, gave a positive solution of the Kurosch problem for SP-algebras, i.e. proved the local finiteness of algebraic SP-algebras of a certain special kind. However, Dresin himself notes the difficulties which did not allow him to solve even Levitzki' s problem for SP-rings without additional restrictions.

If the strictly supporting monomial $\pi(\lambda)$, which in what follows we shall assume to be linear in each variable λ_i, has degree t, then the SP-ring S will be called an SP-ring of degree t.

Lemma. Let S be a nil-SP-ring of degree t, and let I be its ideal generated by the elements a_i^t, where a_i, $i = 1, 2, \dots, n$, is some fixed set of elements of the ring S.

Then for every natural number $q > t$ there exists a natural number $k = k(q)$ such that the ideal I^k belongs to the ideal generated by the elements a_i^q.

Proof. Suppose there exists a natural number r such that the ideal I^r lies in the ideal generated by the elements a_i^m, $i = 1, 2, \dots, n$; $m \geq t$. To prove the lemma we shall show that there exists a natural number r_1 such that the ideal I^{r_1} lies in the ideal generated by the elements a_i^{m+1}.

Every element of the ideal $I^{r(nt+1)}$ can be represented as a sum of products of $nt+1$ elements of the form $\alpha a_i^m \beta$, where α and β are monomials in the generators of the ring S. For each such product there is an element a_j^m occurring in it at least $t+1$ times. Thus each such product can be written in the form

$$c_1 D c_{t+2} = c_1 d_1 d_2 \dots d_{tc_{t+2}} = c_1 (a_j^m c_2 a_j) a_j^{m-1} c_3 a_j^2)(a_j^{m-2} c_4 a_j^3) \dots (a_j^{m-t+1} c_{t+1} a_j^t) c_{t+2}.$$

By assumption, the monomial D belongs to the right ideal generated by all products of its factors, different from D and of degree not less than that of D with respect to the elements d_i.

For any other monomial of degree t in the elements d_i there are two adjacent elements d_{j_1} and d_{j_2} such that $j_1 \geq j_2$. In each such case, in the corresponding segment there will stand the word

$$d_{j_1} d_{j_2} = a_j^{m-j_1+1} c_{j_1+1} a_j^{j_1} a_j^{m-j_2+1} c_{j_2+1} a_j^{j_2} = a_j^{m-j_1+1} c_{j_1+1} a_j^{m+1+(j_1-j_2)} c_{j_2+1} a_j^{j_2}.$$

It is easy to see that all such elements lie in the ideal generated by the element a_j^{m+1}.

It follows that $c_1 D = \omega_1 + c_1 D q$, where ω_1 is an element of the ideal generated by the element a_j^{m+1}. But then

$$c_1 D = \omega_1 + \omega_1 q + c_1 D q^2 = \omega_1 + \omega_1 q + \omega_1 q^2 + c_1 D q^3 = \dots = \omega_1 + \omega_1 q + \omega_1 q^2 + \dots + \omega_1 q^l + c_1 D q^{l+1}$$

for any l. The assertion is proved by virtue of the nilpotency of the element q. As the number r_1 one may take $r(nt+1)$.

Theorem. Every nil-SP-ring is locally nilpotent.

Proof. Let S be a nil-SP-ring of degree t with a finite number of generators, and let J be the ideal generated in it by all possible elements of the form a^t, $a \in S$. The factor ring S/J is nilpotent by Levitzki' s theorem ([5]), and this means that there exists a natural number M such that every element of the form $b_{i_1} b_{i_2} \dots b_{i_M}$, where the b_{i_s} are generators of the ring S, belongs to the ideal J. Since there are only finitely many elements of the form $b_{i_1} b_{i_2} \dots b_{i_M}$, the ideal S^M belongs to some ideal J_1, contained in the ideal J and generated by some finite set of elements of the form a_i^t.

From the lemma it follows that the ideal J_1, and consequently also the ring S, is nilpotent.

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Note: Figure translations are in progress. See original paper for figures.