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Abstract

Full Text

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BOUNDARY PROPERTIES OF DIFFERENTIABLE AND HARMONIC FUNCTIONS IN DOMAINS WITH ANGULAR POINTS

(Presented by Academician S. L. Sobolev, 27 VI 1957)

Consider in space a conical domain G, formed by rotating the ray r about the axis x. Rectangular coordinates are related to spherical ones by the relations

$$z = r\sin\theta\sin\varphi, \qquad y = r\sin\theta\cos\varphi, \qquad x = r\cos\theta,$$

$$0 < r < \infty, \qquad 0 < \theta \le \pi, \qquad 0 < \varphi \le 2\pi.$$

On the boundary Γ of the domain G a function is given,

$$\psi(\rho,\varphi) = \frac{1}{\sqrt{r}} f(\rho,\varphi), \qquad r = e^{\rho}.$$

Let us construct a harmonic function $V(\rho,\varphi,\theta)$ inside the domain G. The Dirichlet integral in the coordinates (ρ,φ,θ) for the function $V(\rho,\varphi,\theta)$ has the form

$$D[V] = \iiint_G \left[\left(\frac{\partial V}{\partial\rho}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial V}{\partial\varphi}\right)^2 + \left(\frac{\partial V}{\partial\theta}\right)^2\right] e^{\rho}\sin\theta\, d\rho\, d\varphi\, d\theta.$$

If the domain G is a cone bounded by two concentric spheres of radii $r = 1$ and $r = e$, then the harmonic function has the form

$$V(\rho,\varphi,\theta) = \frac{1}{\sqrt{e^{\rho}}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sin k\pi\rho\,[A_{km}\cos m\varphi + B_{km}\sin m\varphi]\,\frac{K_k^m(\cos\theta)}{K_k^m(\cos\theta_0)}, \quad (1)$$

where $0 \le \rho \le 1$, $0 < \theta \le \theta_0$, $0 < \varphi \le 2\pi$; $K_k^m(\cos\theta)$ are conical functions (*). A_{km} and B_{km} are the Fourier coefficients of the function

$$f(\rho,\varphi)\sim\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sin k\pi\rho\,[A_{km}\cos m\varphi+B_{km}\sin m\varphi]\,.$$

In the case where the domain G is a full cone with vertex at the origin, symmetric with respect to the axis x, the harmonic function $V(\rho,\varphi,\theta)$ is representable by the expression

$$V(\rho,\varphi,\theta)=\frac{1}{2\pi\sqrt{e^{\rho}}}\sum_{m=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos k(\rho-\rho')\,[a_m(\rho')\cos m\varphi+$$

$$+\,b_m(\rho')\sin m\varphi]\,\frac{K_k^m(\cos\theta)}{K_k^m(\cos\theta_0)}\,dk\,d\rho'. \tag{2}$$

where $-\infty<\rho<\infty$; $0<\theta<\theta_0$; $0<\varphi<2\pi$. Let $A_m(k)$ and $B_m(k)$ denote the Fourier transforms of the coefficients $a_m(\rho)$ and $b_m(\rho)$ of the function

$$f(\rho,\varphi)\sim\sum_{m=0}^{\infty}\{a_m(\rho)\cos m\varphi+b_m(\rho)\sin m\varphi\}.$$

Theorem 1. If a function $V(\rho,\varphi,\theta)$, harmonic in the domain G, satisfies the conditions:

$$1)\qquad \iint_{\Gamma}|V(\rho,\varphi,\theta)|^2\,d\Gamma\leqslant N\quad\text{for all }\theta<\theta_0;$$

$$2)\quad D[V]<\infty,$$

then there exists a boundary function $\psi(\rho,\varphi)$, summable in L_2, for which

$$\lim_{\theta\to\theta_0}\iint_{\Gamma}|V(\rho,\varphi,\theta)-\psi(\rho,\varphi)|^2\,d\Gamma=0$$

and the condition

$$\iint_{\Gamma}|\psi(\rho+h_1,\varphi+h_2)-\psi(\rho-h_1,\varphi-h_2)|^2\,d\Gamma\leqslant M\sum_{k=1}^{2}h_k^{1+\varepsilon}$$

is fulfilled, where Γ is the boundary of the domain G; $\varepsilon=0$; M is a constant independent of h_k.

Theorem 2. If the boundary function $\psi(\rho,\varphi)$ is summable in L_2 and satisfies the condition

$$\iint_{\Gamma} |\psi(\rho + h_1, \varphi + h_2) - \psi(\rho - h_1, \varphi - h_2)|^2 \, d\Gamma \leqslant M \sum_{k=1}^{2} h_k^{1+\varepsilon}, \tag{3}$$

where $\varepsilon > 0$, then the harmonic function in the domain G corresponding to it has the following properties:

$$1) \quad V(\rho, \varphi, \theta) \text{ is harmonic in } G \text{ for } \theta < \theta_0;$$

$$2) \quad \iint_{\Gamma} |V(\rho, \varphi, \theta)|^2 \, d\Gamma \leqslant N \quad \text{for } \theta < \theta_0;$$

$$3) \quad \lim_{\theta \to \theta_0} \iint_{\Gamma} |V(\rho, \varphi, \theta) - \psi(\rho, \varphi)|^2 \, d\Gamma = 0;$$

$$4) \quad D[V] < \infty.$$

Theorem 3. If the boundary function $\psi(\rho, \varphi)$ is summable in L_2 and satisfies the condition

$$\iint_{\Gamma} |\psi(\rho + h_1, \varphi + h_2) - \psi(\rho - h_1, \varphi - h_2)|^2 \, d\Gamma \leqslant M_1 \sum_{k=1}^{2} h_k^{2\alpha},$$

then, for the corresponding harmonic function $V(\rho, \varphi, \theta)$ in the domain G,

$$\iiint_{G} |V(\rho + h_1, \varphi + h_2, \theta + h_3) - V(\rho - h_1, \varphi - h_2, \theta - h_3)|^2 \, dG \leqslant M_2 \sum_{k=1}^{3} h_k^{2\alpha+1},$$

where M_1 and M_2 are constants independent of h_k.

In the proof of Theorems 1 and 2 it is shown that, if $V(\rho, \varphi, \theta)$ is representable by expression (1), then the finiteness of $D[V]$ is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (k + m)(A_{km}^2 + B_{km}^2). \tag{4}$$

From condition (3) of Theorem 2, the convergence of the series (4) is proved. If $V(\rho, \varphi, \theta)$ is represented by formula (2), then the finiteness of $D[V]$ is equivalent to the convergence of the series

$$\sum_{m=0}^{\infty}\int_{-\infty}^{\infty}(m+k)'\,[A_m^2(k)+B_m^2(k)]\,dk. \tag{5}$$

It is shown that, from condition (3) of Theorem 2, in this case the convergence of the series (5) follows.

In the proof of Theorem 3, the idea of the proof of a theorem of S. N. Bernstein ([4]) is used.

Consider the domain $G:\ \ 0 \leq x,y,z \leq \pi$. A function $f(x,y,z) \in H_P^r(G)$ induces on the boundary Γ of the domain G the function $f|_\Gamma = \varphi$. In this case the following theorems are proved:

Theorem 4. If $0 < r - 1/P < 1$ and the function f belongs to the class $H_P^r(G)$, then the boundary function $\varphi \in H_P^{r-1/P}(\overline{M})$, where $\overline{M} < C\|f\|^r_{L_P(G)}$; C is a constant independent of $\|f\|^r_{L_P(G)}$.

Theorem 5. If $0 < r - 1/P < 1$ and on the boundary Γ a function φ is given which belongs to the class $H_P^{r-1/P}(M)$, then there exists a function f of the class $H_P^r(G)$, defined in the domain G, such that $f|_\Gamma = \varphi$,

$$\|f\|^r_{L_P(G)} < C\|\varphi\|^r_{L_P(\Gamma)}.$$

Under the imposition of a continuity condition on the even derivatives of the function φ at the corner points, Theorems 4 and 5 are proved in the case $r - 1/P > 1^*$.

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* For the definition of $H_P^r(G)$, $\|f\|^r_{L_P(G)}$, $\|\varphi\|^r_{L_P(\Gamma)}$, see ([3]).

Note: Figure translations are in progress. See original paper for figures.