ON THE QUESTION OF NONSTATIONARY PLANE FLOWS OF A POLYTROPIC GAS WITH RECTILINEAR CHARACTERISTICS

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Abstract

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ON THE QUESTION OF NONSTATIONARY PLANE FLOWS OF A POLYTROPIC GAS WITH RECTILINEAR CHARACTERISTICS

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In the present note a complete classification is given of unsteady plane flows of a polytropic gas having rectilinear characteristics–common level lines of the quantities u_i, c in the phase space x_1, x_2, t.

The equations of hydrodynamics for our case have the form: the Euler equations

$$\frac{\partial u_i}{\partial t} + 2\kappa c \frac{\partial c}{\partial x_i} + \sum_{k=1}^{2} u_k \frac{\partial u_i}{\partial x_k} = 0, \qquad i = 1,\ 2; \tag{1}$$

the continuity equation

$$2\kappa \left(\frac{\partial c}{\partial t} + \sum_{k=1}^{2} u_k \frac{\partial c}{\partial x_k} \right) + c \sum_{k=1}^{2} \frac{\partial u_k}{\partial x_k} = 0, \tag{2}$$

where $\kappa = \dfrac{1}{\gamma - 1}$; $\gamma = \dfrac{c_p}{c_v} > 1$ is the adiabatic exponent; $c^2 = \left(\dfrac{dp}{d\rho}\right)_S$; $p = a^2(S)\rho^\gamma$; S is the entropy.

We shall specify the equations of the characteristics in the form

$$\frac{dx_1}{\Delta_1} = \frac{dx_2}{\Delta_2} = \frac{dt}{1}, \tag{3}$$

and, since we are considering rectilinear characteristics, the functions u_i, c, Δ_i will be regarded as depending on two parameters α_1, α_2, which we take to be the quantities

$$x_i - \Delta_i t = \alpha_i, \qquad i = 1,\ 2. \tag{4}$$

Differentiating (4) with respect to x_k and t, we find expressions for $\partial\alpha_i/\partial x_k$ and $\partial\alpha_i/\partial t$, and then, using the expressions found, pass in equations (1), (2) to the variables α_1, α_2. After this they can be written in the form:

$$A_i + tB_i = 0, \qquad i = 1, 2, 3, \tag{5}$$

where

$$A_1 = \frac{2}{\gamma - 1}\frac{\partial c}{\partial \alpha_1} + l_1\frac{\partial u_1}{\partial \alpha_1} + l_2\frac{\partial u_1}{\partial \alpha_2},$$

$$A_2 = \frac{2}{\gamma - 1}\frac{\partial c}{\partial \alpha_2} + l_1\frac{\partial u_2}{\partial \alpha_1} + l_2\frac{\partial u_2}{\partial \alpha_2},$$

$$A_3 = \frac{2}{\gamma - 1}l_1\frac{\partial c}{\partial \alpha_1} + \frac{2}{\gamma - 1}l_2\frac{\partial c}{\partial \alpha_2} + \frac{\partial u_1}{\partial \alpha_1} + \frac{\partial u_2}{\partial \alpha_2},$$

$$B_1 = \frac{2}{\gamma - 1}p_{22}\frac{\partial c}{\partial \alpha_1} - \frac{2}{\gamma - 1}p_{21}\frac{\partial c}{\partial \alpha_2} + (l_1 p_{22} - l_2 p_{12})\frac{\partial u_1}{\partial \alpha_1} + (l_2 p_{11} - l_1 p_{21})\frac{\partial u_1}{\partial \alpha_2},$$

$$B_2 = -\frac{2}{\gamma - 1}p_{12}\frac{\partial c}{\partial \alpha_1} + \frac{2}{\gamma - 1}p_{11}\frac{\partial c}{\partial \alpha_2} + (l_1 p_{22} - l_2 p_{12})\frac{\partial u_2}{\partial \alpha_1} + (l_2 p_{11} - l_1 p_{21})\frac{\partial u_2}{\partial \alpha_2},$$

$$B_3 = \frac{2}{\gamma - 1}(l_1 p_{22} - l_2 p_{12})\frac{\partial c}{\partial \alpha_1} + \frac{2}{\gamma - 1}(l_2 p_{11} - l_1 p_{21})\frac{\partial c}{\partial \alpha_2} + p_{22}\frac{\partial u_1}{\partial \alpha_1}$$
$$-p_{21}\frac{\partial u_1}{\partial \alpha_2} - p_{12}\frac{\partial u_2}{\partial \alpha_1} + p_{11}\frac{\partial u_2}{\partial \alpha_2},$$

$$p_{ij} = \frac{\partial \Delta_i}{\partial \alpha_j}, \qquad l_i = \frac{u_i - \Delta_i}{c}.$$

Since A_i and B_i are functions of α_1, α_2, while equations (5) must be satisfied for arbitrary t, the conditions

$$A_i = 0, \qquad B_i = 0, \qquad i = 1,\ 2,\ 3 \tag{6}$$

must hold.

We have obtained an overdetermined system of 6 equations for 5 unknown functions, whose compatibility is to be investigated. At first we shall assume that u_1 and u_2 are functionally independent, i.e.

$$K = \begin{vmatrix} \dfrac{\partial u_1}{\partial \alpha_1} & \dfrac{\partial u_1}{\partial \alpha_2} \\ \dfrac{\partial u_2}{\partial \alpha_1} & \dfrac{\partial u_2}{\partial \alpha_2} \end{vmatrix} \neq 0. \tag{7}$$

Putting $c = c(u_1, u_2)$ and introducing the function

$$\theta(u_1, u_2) = \frac{2}{\gamma - 1} c(u_1, u_2), \tag{8}$$

the system of equations (6) can be written as a system of 6 equations homogeneous with respect to the derivatives $\partial u_i / \partial \alpha_k$. The rank of this system is $r \leq 3$.

Investigating this system, we arrive at the following results:

1. The case $r = 1$ is trivial; it leads to flows with $c = \text{const}$.
2. The case $r = 2$ gives the following possibilities:

a) Conical flows characterized by the conditions

$$p_{21} = p_{12} = 0, \qquad p_{11} = p_{22}. \tag{9}$$

b) Potential flows, with the equations $B_1 = 0,\ B_2 = 0$ satisfied automatically, while the equation $B_3 = 0$ gives a second-order equation for the function θ:

$$\frac{\gamma - 1}{2} \theta \left[(1 - \theta_1^2)\theta_{22} + 2\theta_1\theta_2\theta_{12} + (1 - \theta_2^2)\theta_{11}\right] + \frac{\gamma - 3}{2}(\theta_1^2 + \theta_2^2) + 2 = 0, \tag{10}$$

$$\theta_i = \frac{\partial \theta}{\partial u_i}, \qquad \theta_{ik} = \frac{\partial^2 \theta}{\partial u_i \partial u_k};$$

Δ_i are related to θ by the relations

$$\Delta_i = u_i + \frac{\gamma - 1}{2} \theta \theta_i; \tag{11}$$

u_1 and u_2 are determined from the equations $A_2 = 0,\ A_3 = 0$, and then are found as functions of x_1, x_2, t from the relations (4).

c) In the case $\gamma = 2$, and only in this case, there exist vortical flows described by the equations

$$2\theta^2 = (2u_1 + A)^2 + (2u_2 + B)^2 \tag{12}$$

(A and B are constants);

$$(x + y)\frac{\partial \alpha_2}{\partial y} + x\frac{\partial \alpha_1}{\partial y} - y\frac{\partial \alpha_2}{\partial x} = 0, \tag{13}$$

$$x\frac{\partial \alpha_1}{\partial y} + y\frac{\partial \alpha_2}{\partial x} + (x - y)\frac{\partial \alpha_1}{\partial x} = 0,$$

where $2u_1 + A = x,\ 2u_2 + B = y$; Δ_i are determined from the equalities

$$l_i = (-1)^{i+1} \frac{\theta_{3-i}}{\sqrt{\theta_1^2 + \theta_2^2 - 1}}. \tag{14}$$

Let us indicate one particular exact solution of system (13). Let $y/x = \xi$. Then the functions

$$\alpha_1 = a_1 \ln x + \tilde{\alpha}_1(\xi),$$

$$\alpha_2 = a_2 \ln x + \tilde{\alpha}_2(\xi), \tag{15}$$

where

$$\tilde{\alpha}_1 = \int_0^\xi -\frac{\xi(1+\xi)a_2 + (1-\xi^3)a_1}{(1+\xi^2)^2}\, d\xi + c_1,$$

$$\tilde{\alpha}_2 = \int_0^\xi \frac{a_1(1-\xi) + 2a_2\xi - \xi^2(1-\xi)a_2}{(1+\xi^2)^2}\, d\xi + c_2; \tag{16}$$

a_1, a_2, c_1, c_2 are arbitrary constants, are exact solutions of system (14), and the corresponding flow is, generally speaking, vortical and belongs neither to the class of simple waves nor to the class of conical flows.

3. The case $r = 3$ leads only to conical flows, described by the equations $A_i = 0,\ i = 1, 2, 3$, and by conditions (9).

Considering further the case when $K = 0$ (7), for example $u_2 = f(u_1)$, and investigating the system of 6 equations homogeneous with respect to $\dfrac{\partial \theta}{\partial \alpha_i}, \dfrac{\partial u_1}{\partial \alpha_i}$, obtained from system (6), in complete analogy with the preceding case one can show that in the given class of flows there enter only simple waves or conical flows.

Thus all cases have been investigated, and the following statement may be formulated:

In the class of nonstationary plane adiabatic motions of a gas ($\gamma \neq 2$) with rectilinear characteristics, there do not exist vortical flows distinct from simple waves and conical flows.

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Note: Figure translations are in progress. See original paper for figures.