# MATHEMATICS

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**Abstract**

**Full Text**

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# POLYNOMIAL OPERATIONS IN CERTAIN SPACES

*(Presented by Academician A. N. Kolmogorov, 19 VIII 1957)*

I. Consider a linear normed space $E$ for which the following axioms are satisfied:

1. The elements of $E$ are $2\pi$-periodic real functions of a real argument $x$, summable on $[0, 2\pi]$.
2. Addition of elements of $E$ and multiplication of an element by a real number are defined in the usual way.
3. The space $E$ contains all trigonometric polynomials (the set of these polynomials will be denoted by $T$, and the set of all trigonometric polynomials of order $\leq n$ by $T_n$).
4. Membership of an element in $E$ and the norm of an element are invariant with respect to an arbitrary shift of the argument, i.e., from $f(x) \in E$ it follows that $f(x+t) \in E$ and $\|f(x+t)\| = \|f(x)\|$ for any $t$.
5. The set $T$ is everywhere dense in $E$ in the sense of convergence in the norm (strong convergence).

Any space possessing these properties will be called, for brevity, a **space of type** $E$. The simplest examples are the spaces $\tilde{C}$ and $\tilde{L}^p$ $(p \geq 1)$ of $2\pi$-periodic functions (with the usual definition of the norm).

II. Let $E_1$ and $E_2$ be spaces of type $E$. A linear operation $A_n(f, x)$ from $E_1$ to $E_2$ is called a **trigonometric polynomial operation of order** $n$ (briefly, a polynomial operation) if it maps the space $E_1$ into the subset $T_n$ of the space $E_2$.

For the norm $\|A_n\|$ of the linear operation $A_n(f, x)$ from $E_1$ to $E_2$ the notation $\|A_n\|_{E_1}^{E_2}$ will also be used.

III. Let $M$ be some subset of $E_1$. An operation $U(f, x)$ from $E_1$ to $E_2$ is called **sliding on the set** $M$ if, for each element $f(x)$ of the set $M$, for any $t$ the relation $U(f(x+t), x) = U(f(x), x+t)$ or $U(f(x+t), x-t) = U(f(x), x)$ holds.

For example, the operation

$$B_n(f,x) = \int_0^{2\pi} K_n(t) f(x+t)\,dt,$$

where $K_n(x)$ is a fixed element of $T_n$, is sliding on the whole set $E_1$. An operation identical on $M$ (i.e., leaving the elements of $M$ unchanged) is sliding on $M$.

IV. **The fundamental formula for a trigonometric polynomial operation.** For every linear operation $A_n(f,x)$ from $E_1$ to $E_2$, mapping $E_1$ onto $T_n$, the formula

$$\int_0^{2\pi} A_n(f(x+t), x-t)\,dt = \int_0^{2\pi} A_n(S_n(x+t), x-t)\,dt, \tag{1}$$

is valid for any value of $x$,

where $f(x)$ is any element of $E_1$; $S_n(x) = S_n(f,x)$ is the $n$-th partial sum of the Fourier series of the function $f(x)$.

**Particular cases.** 1) If the operation $A_n(f,x)$ is a sliding one on the whole set $E_1$ (we denote this operation by $B_n(f,x)$), then

$$B_n(f,x) = B_n(S_n, x). \tag{2}$$

It follows from this that every sliding polynomial operation from $E_1$ to $E_2$ is representable in the form

$$B_n(f,x) = \int_0^{2\pi} K_n(t) f(x+t)\,dt, \tag{3}$$

where $K_n(x)$ is a fixed element of $T_n$ (namely, $K_n(x) = B_n\left(\frac{1}{\pi} D_n(x), -x\right)$).

2) If the operation $A_n(f,x)$ coincides, on each element of the set $T_n$, with the sliding operation (from $E_1$ to $E_2$) $B_n(f,x)$, then

$$B_n(f,x) = \frac{1}{2\pi}\int_0^{2\pi} A_n(f(x+t), x-t)\,dt. \tag{4}$$

In particular, if $A_n(f,x)$ coincides on $T_n$ with the operation $S_n(f,x)$, i.e. if $A_n(f,x)$ leaves the elements of $T_n$ unchanged, then

$$S_n(f,x) = \frac{1}{2\pi}\int_0^{2\pi} A_n(f(x+t), x-t)\,dt. \tag{5}$$

If the operation $A_n(f,x)$ takes each element $\tau_n(x)$ of the set $T$ into the conjugate polynomial $\tilde{\tau}_n(x)$, then

$$\widetilde{S}_n(f,x) = \frac{1}{2\pi}\int_0^{2\pi} A_n(f(x+t), x-t)\,dt. \tag{6}$$

V. **Extremal properties of the sliding polynomial operation.** 1) Among all polynomial operations $A_n(f,x)$ from $E_1$ to $E_2$ that coincide on the set $T_n$ with a given polynomial operation $B_n(f,x)$ sliding on all of $E_1$, the operation $B_n(f,x)$ has the least norm, i.e.

$$\|B_n\| \leqslant \|A_n\|. \tag{7}$$

In particular, among all polynomial operations $A_n(f,x)$ that preserve the set $T_n$, the operation $S_n(f,x)$ has the least norm, i.e.

$$\|S_n\| \leqslant \|A_n\|. \tag{8}$$

Analogously for the operation occurring in (6).

2) Let $A_n(f,x)$ be a polynomial operation from $E$ to $E$, coinciding on $T_n$ with the sliding polynomial operation $B_n(f,x)$ (from $E$ to $E$). Let $N(f)$ be a positive functional on $E$ (not necessarily linear) having the property that $N(f(x+t)) = N(f(x))$ for any $t$. If $M$ is an arbitrary bounded subset of $E$, moreover such that $f(x) \in M$ implies $f(x+t) \in M$, then the inequality holds

$$\sup_{f\in M} \frac{\|f(x) - B_n(f,x)\|}{N(f)} \leqslant \sup_{f\in M} \frac{\|f(x) - A_n(f,x)\|}{N(f)}. \tag{9}$$

3) We shall consider the norms of the sliding operation $B_n(f,x)$ from $E$ to $E$ for all possible spaces of type $E$; the inequality holds

$$\|B_n\|_E^E \leq \|B_n\|_{\widetilde{C}}^{\widetilde{C}} = \int_0^{2\pi} |K_n(t)|\,dt. \tag{10}$$

In particular, $\|S_n\|_E^E \leq \|S_n\|_{\widetilde{C}}^{\widetilde{C}} = L_n$, where $L_n$ is the Lebesgue constant for the ordinary Fourier series.

**Remark.** The results listed above are in part adjacent to the results of the works of S. M. Lozinskii ([1]) and contain as a special case the results of the papers ([2]) of D. L. Berman, who assumed, with respect to the space $E$, besides axioms 1-5 of the present article, also a certain additional assumption on the structure of the norm in $\widetilde{E}$.

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## CITED LITERATURE

[1] S. M. Lozinskii, DAN, **61**, No. 2 (1948); **64**, No. 4 (1949); **89**, No. 4 (1953); **89**, No. 5 (1953). [2] D. L. Berman, DAN, **88**, No. 1 (1952); **91**, No. 6 (1953); **92**, No. 4 (1953); **95**, No. 2 (1954).

*Note: Figure translations are in progress. See original paper for figures.*