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Abstract

Full Text

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ON THE CAUCHY PROBLEM FOR QUASILINEAR EQUATIONS

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For the consideration of discontinuous solutions of conservative systems of quasilinear equations ([1]), the concept of a generalized solution is introduced. A **generalized solution of the Cauchy problem for a system of equations**

$$\frac{\partial u_i}{\partial t} + \frac{\partial \varphi_i(u_1, u_2, \dots, u_n, t, x)}{\partial x} = 0 \quad (i = 1, 2, \dots, n) \tag{1}$$

is understood to be such functions $u_i(t, x)$ that take the prescribed values on the initial line $t = 0$ and satisfy the integral relations:

$$\oint_C u_i(t, x)\, dx - \varphi_i(u, t, x)\, dt = 0 \quad (i = 1, 2, \dots, n), \tag{2}$$

where $u = \{u_1(t, x); u_2(t, x); \dots; u_n(t, x)\}$; C is any piecewise-smooth closed contour lying entirely in the domain $t \geq 0$.

It is known that conditions (2) do not, in general, guarantee uniqueness of the generalized solution of the Cauchy problem for system (1). Therefore one requires of a generalized solution of the system of equations (1) that it satisfy certain additional conditions. In the case of one quasilinear equation these conditions were formulated in ([2]), and in the case of n equations–in ([3]).

The question of uniqueness of the generalized solution of the Cauchy problem for one quasilinear equation is the subject of the works ([2], [4]). For some special cases of a system of quasilinear equations of the form (1), this question was considered in the works ([5], [6]).

Here we indicate a method that makes it possible to reduce the question of uniqueness of the generalized (discontinuous) solution of the Cauchy problem for the system of quasilinear equations (1) to the question of uniqueness of the continuous solution of the Cauchy problem for a certain system of nonlinear equations. Such a reduction makes it possible to prove a uniqueness theorem for the solution of a system of quasilinear equations. In the present note we shall illustrate this method by the example of proving uniqueness of the generalized solution of the Cauchy problem for one quasilinear equation.

1. Suppose that the functions $\varphi_i(u, t, x)$ are bounded, continuous, and continuously differentiable with respect to all their arguments in the domain under consideration of variation of the variables u, t, x. Let the system of functions $u(t, x) = \{u_1(t, x); u_2(t, x); \dots; u_n(t, x)\}$ be a generalized solution of the system of quasilinear equations (1), piecewise continuous for $t > 0$. Let us introduce into consideration the functions $\Phi_i(t, x)$, defined by means of the equations:

$$\Phi_i(t, x) = \int_{(0,0)}^{(t,x)} u_i(t, x)\, dx - \varphi_i(u(t, x), t, x)\, dt \quad (i = 1, 2, \dots, n). \tag{3}$$

By virtue of relations (2), the functions $\Phi_i(t, x)$ are continuous functions of the variables t, x. At those points of the half-plane $t \geqslant 0$ at which the functions $u_i(t, x)$ are continuous, the equalities hold

$$\frac{\partial \Phi_i(t, x)}{\partial x} = u_i(t, x); \qquad \frac{\partial \Phi_i(t, x)}{\partial t} = -\varphi_i(u, t, x) \quad (i = 1, 2, \dots, n). \tag{4}$$

Eliminating from equations (4) the variables $u_i(t, x)$, we obtain a system of nonlinear equations satisfied by the functions $\Phi_i(t, x)$:

$$\frac{\partial \Phi_i}{\partial t} + \varphi_i\left(\frac{\partial \Phi}{\partial x}, t, x\right) = 0 \quad (i = 1, 2, \dots, n), \tag{5}$$

where $\Phi = \{\Phi_1, \dots, \Phi_n\}$; $\dfrac{\partial \Phi}{\partial x} = \left\{\dfrac{\partial \Phi_1}{\partial x}, \dots, \dfrac{\partial \Phi_n}{\partial x}\right\}$.

If there exists a generalized solution of the Cauchy problem for the system of quasilinear equations (1), piecewise continuous for $t > 0$, then there exists a continuous solution of the system of equations (5). The converse assertion is also valid: if there exists a piecewise smooth continuous solution of the system of nonlinear equations (5) satisfying the conditions

$$\Phi_i(0, x) = \int_0^x u_i(0, \tau)\, d\tau, \tag{6}$$

then $u_i(t, x) = \partial \Phi_i / \partial x$ is a generalized solution of the Cauchy problem for the system of equations (1).

We shall call the system of functions $\Phi = \{\Phi_1, \dots, \Phi_n\}$, satisfying equations (5) and the initial conditions (6), the **potentials of a generalized solution of the system of quasilinear equations** (1).

We shall say that two generalized solutions of the Cauchy problem for the system of equations (1) **coincide** if they coincide at all points where they are continuous.

It follows from this that, if the potentials of two generalized solutions of the Cauchy problem for the system of quasilinear equations (1) coincide with one another, then the generalized solutions themselves also coincide.

Thus, we have reduced the Cauchy problem for the system of quasilinear equations (1) in the class of generalized (discontinuous) solutions to the Cauchy problem for the system of nonlinear equations (5) in the class of piecewise smooth continuous solutions.

Thereby the question of uniqueness of the generalized solution of the Cauchy problem for the system (1) is reduced to the question of uniqueness of the solution of the system of equations (6).

2. Consider one quasilinear equation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u,t,x)}{\partial x} = 0 \tag{7}$$

and suppose that $\varphi''_{uu}(u,t,x)$ is continuous and bounded in the whole domain under consideration of the variables u, t, x. Let

$$\varphi''_{uu}(\bar{u},t,x)\varphi''_{uu}(\bar{\bar{u}},t,x) \geqslant 0 \tag{8}$$

for any two values $\bar{u}$ and $\bar{\bar{u}}$ from the domain under consideration.

Theorem. A generalized solution of the Cauchy problem for equation (7), piecewise continuous for $t > 0$, satisfying the condition

$$\varphi'_u(u(t,x+0),t,x) \leqslant \varphi'_u(u(t,x-0),t,x) \tag{9}$$

for all points (t,x) from the domain of definition of the solution, is unique.

Proof. Suppose the contrary. Let $\bar{u}(t,x)$ and $\bar{\bar{u}}(t,x)$ be two distinct generalized solutions of equation (7), taking the same initial···

initial values at $t = 0$. They correspond to two continuous functions $\overline{\Phi}(t,x)$ and $\overline{\overline{\Phi}}(t,x)$, taking the same values at $t = 0$, which are potentials of the corresponding solutions. The difference of the potentials $v(t,x) = \overline{\Phi}(t,x) - \overline{\overline{\Phi}}(t,x)$ satisfies the equation

$$\frac{\partial v}{\partial t} + A(t,x)\frac{\partial v}{\partial x} = 0, \tag{10}$$

where

$$A(t,x)=\begin{cases}\dfrac{\varphi(\overline{u}(t,x),t,x)-\varphi(\overline{\overline{u}}(t,x),t,x)}{\overline{u}(t,x)-\overline{\overline{u}}(t,x)}, & \text{if } \overline{u}(t,x)\neq\overline{\overline{u}}(t,x),\\ \varphi_u(\overline{u}(t,x),t,x), & \text{if } \overline{u}(t,x)=\overline{\overline{u}}(t,x).\end{cases} \tag{11}$$

Let us note that, in view of the restrictions imposed on $\varphi(u,t,x)$, the function $A(t,x)$ has discontinuities only at those points at which $\overline{u}(t,x)$ and $\overline{\overline{u}}(t,x)$ have discontinuities.

To prove the theorem it is enough to show that $v(t,x)\equiv 0$. In view of conditions (6), $v(0,x)\equiv 0$. Consider the ordinary differential equation of first order:

$$\frac{dx}{dt}=A(t,x). \tag{12}$$

The integral curves of equation (12) are the characteristics of the linear equation (10), and on each of them the function $v(t,x)$ is constant. Therefore it is enough to show that through any point (t,x) of the half-plane $t>0$ there passes at least one integral curve of equation (12) intersecting the axis $t=0$.

Definition 1. We shall call a point (t_0,x_0) a **regular point of equation** (12) if in some neighborhood of this point there exists an integral curve of equation (12), defined both for $t>t_0$ and for $t\le t_0$, passing through the point (t_0,x_0).

Definition 2. We shall call a point (t_0,x_0) a **return point of an integral curve of equation** (12) if in a neighborhood of this point there is no integral curve of equation (12) passing through the point (t_0,x_0) and defined for $t>t_0$, but there exists an integral curve of equation (12), defined for $t\le t_0$, passing through the given point.

Definition 3. We shall call a point (t_0,x_0) a **branch point for equation** (12) if for $t<t_0$ there is no integral curve passing through this point.

Lemma 1. Points (t_0,x_0) which are points of continuity of $\overline{u}(t,x)$, $\overline{\overline{u}}(t,x)$ (and therefore of $A(t,x)$) are regular points of equation (12).

Lemma 2. An integral curve of equation (12) issuing from a regular point (t_0,x_0) necessarily intersects the axis $t=0$, provided that on its path it does not meet a branch point of equation (12).

Lemma 3. If all points (t,x) of the half-plane $t>0$ are either regular points or return points of equation (12), then through every point of the half-plane $t>0$ there passes at least one integral curve of equation (12) intersecting the axis $t=0$.

In this case $v(t,x)\equiv 0$. Thus, it remains for us to show that, under the conditions formulated in the theorem, equation (12) has no branch points.

We shall henceforth consider only points of discontinuity of the function $A(t,x)$ lying on certain lines. If $A(t,x)$ has an isolated discontinuity point (t_0,x_0), then, since in any of its neighborhoods there are regular points of equation (12), $v(t_0, x_0) = 0$ by the continuity of the function $v(t, x)$.

Lemma 4. If (t_0, x_0) is a branching point of equation (12), then

$$A(t_0, x_0 + 0) > A(t_0, x_0 - 0). \tag{13}$$

Thus, we have to show that inequality (13) cannot hold. Then the theorem will be proved.

Let the point (t_0, x_0) be a point of discontinuity of the function $A(t, x)$. Hence, at this point $\bar{u}(t, x)$ and, possibly, $\bar{\bar{u}}(t, x)$ have a discontinuity.

We have the equalities:

$$A(t_0, x_0 - 0) = \frac{\varphi(\bar{u}(t_0, x_0 - 0), t_0, x_0) - \varphi(\bar{\bar{u}}(t_0, x_0 - 0), t_0, x_0)}{\bar{u}(t_0, x_0 - 0) - \bar{\bar{u}}(t_0, x_0 - 0)},$$

$$A(t_0, x_0 + 0) = \frac{\varphi(\bar{u}(t_0, x_0 + 0), t_0, x_0) - \varphi(\bar{\bar{u}}(t_0, x_0 + 0), t_0, x_0)}{\bar{u}(t_0, x_0 + 0) - \bar{\bar{u}}(t_0, x_0 + 0)}. \tag{14}$$

But, according to (9),

$$\varphi'_u(\bar{u}(t_0, x_0 + 0), t_0, x_0) \le \varphi'_u(\bar{u}(t_0, x_0 - 0), t_0, x_0),$$

$$\varphi'_u(\bar{\bar{u}}(t_0, x_0 + 0), t_0, x_0) \le \varphi'_u(\bar{\bar{u}}(t_0, x_0 - 0), t_0, x_0), \tag{15}$$

whence, by virtue of condition (8), we obtain that

$$A(t_0, x_0 + 0) \le A(t_0, x_0 - 0). \tag{16}$$

Inequality (16) shows that all points of discontinuity of the function $A(t, x)$ are either regular points or return points of equation (12). The theorem is proved.

The method set forth for proving the uniqueness theorem for a generalized solution of the Cauchy problem is quite general and is applicable both to quasi-linear and to linear equations of hyperbolic type. By means of potentials of a generalized solution, we have obtained a proof of the uniqueness theorem for a generalized solution of the Cauchy problem also for systems of quasilinear equations of the form (1), under certain assumptions concerning the functions $\varphi_i(u, t, x)$.

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Note: Figure translations are in progress. See original paper for figures.