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Abstract

Full Text

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On Perfect Compact Kernels of Sets Effectively Distinct from All Φ-Sets

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A number of fundamental problems of descriptive set theory are known, whose investigation has led to the boundaries of applicability of set-theoretic principles. Among these problems we shall point to the continuum problem and to the problem of the existence of a perfect compact kernel in projective sets.

With regard to the first of these problems, K. Gödel proved that Cantor' s continuum hypothesis does **not** contradict the system of axioms of set theory. As for the second problem, as P. S. Novikov showed, the assertion of the existence of a perfect compact kernel in projective sets is **not derivable** logically from set-theoretic principles. But since the question of the opposite assertions has not been resolved, N. N. Luzin' s supposition on the unsolvability of the above-mentioned problems of descriptive set theory has not yet been confirmed.

In this state of affairs it is of interest to consider the same problems of descriptive set theory in a formulation restricted by means of the notion of effective distinction, due to P. S. Novikov ([1−3]). P. S. Novikov proved that the continuum problem in its effectively restricted formulation is resolved in the direction of confirming Cantor' s hypothesis. But, as shown in ([3]) and in the present paper, the proposition on the existence of a perfect compact kernel in the sets T_Φ, effectively distinct from projective sets, is derivable from set-theoretic principles.

Here we consider this result for a very general class of δs-operations Φ, which includes all operations yielding B-sets, C-sets, and projective sets of each given class (in item 1 beginning with F_σ, and in item 2 beginning with $F_{\sigma\delta}$). At the same time it is not even required that T_Φ be the complement of some Φ-set. Moreover, the complements CT_Φ also contain perfect compact kernels. The exposition is based on § 4 of ([3]).

1°. By Φ in this item is meant an arbitrary δs-operation possessing the following property (τ): whatever the number n_0 may be, there exists such a (finite or infinite) chain $\{n_1, n_2, ...\}$ of operations Φ that

$$n_0 < n_1 < n_2 < \cdots .$$

R is an arbitrary metric space in which there exist sets effectively distinct from all Φ-sets ([3], §6).

Theorem 1. *Let the set T ($T \subset R$) be effectively distinct from all Φ-sets of the space R. Whatever the Φ-set M may be, $M \subset T$, there exists a discontinuum D which is contained in T and does not meet M: $D \subset T \cdot (R - M)$.*

Proof. Let $M \cdot Z = M_0 = \Phi\{F_n^0\}$, where $\{F_n^0\} \in \Pi^*(Z)$, and let

$$\nu\{F_n^0\} = x_0.$$

Then

$$x_0 \in T \cdot CM_0 + CT \cdot M_0 = T \cdot CM_0.$$

Denoting

$F_n^0 + \langle x_0 \rangle = F_n^1$ and $\nu\{F_n^1\} = x_1$, we obtain $\Phi\{F_n^1\} = \Phi\{F_n^0\} + \langle x_0 \rangle \subset T$, $x_1 \in T \cdot C\Phi\{F_n^1\}$, $x_0 \neq x_1$.

Surround the points x_0 and x_1 by such ε-neighborhoods, respectively S_0 and S_1, that $\varepsilon < 1/2$ and $\overline{S}_0 \cdot \overline{S}_1 = 0$. From the continuity of ν on $\Pi t(Z)$ there follows the existence of a number q_1 such that

$$\nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, Q_1, Q_2, \dots, Q_n, \dots\} \in S_{t_1}$$

($t_1 = 0, 1$), whatever the point $\{Q_n\} \in \Pi t(Z)$. We next introduce the notation:

$$F_n^{t_1 0} = F_n^0, \quad \nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, F_{q_1+1}^{t_1 0}, F_{q_1+2}^{t_1 0}, \dots, F_n^{t_1 0}, \dots\} = x_{t_1 0},$$

$$F_n^{t_1 1} = F_n^{t_1 0} + \langle x_{t_1 0} \rangle, \quad \nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, F_{q_1+1}^{t_1 1}, F_{q_1+2}^{t_1 1}, \dots, F_n^{t_1 1}, \dots\} = x_{t_1 1}.$$

Hence, by virtue of the property of the function ν and the property (τ) of the operation Φ, we obtain:

$$x_{t_1 1} \in T \cdot C\Phi\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, F_{q_1+1}^{t_1 1}, F_{q_1+2}^{t_1 1}, \dots, F_n^{t_1 1}, \dots\},$$

$$x_{t_1 0} \in \Phi\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, F_{q_1+1}^{t_1 1}, F_{q_1+2}^{t_1 1}, \dots, F_n^{t_1 1}, \dots\},$$

$x_{t_1 0} \neq x_{t_1 1}$. This makes it possible to surround $x_{t_1 0}$ and $x_{t_1 1}$ by such ε-neighborhoods, respectively $S_{t_1 0}$ and $S_{t_1 1}$, that

$$S_{t_1 t_2} \subset S_{t_1}, \quad \overline{S}_{t_1 0} \cdot \overline{S}_{t_1 1} = 0, \quad \varepsilon < 1/4.$$

Suppose now that for $m \geq 2$: 1) integers $q_0, q_1, \dots, q_{m-1}$ have been constructed, with $q_0 = 0 < q_1 < \dots < q_{m-1}$; 2) for each of the tuples

$$t_1, \ t_1 t_2, \dots, t_1 t_2 \dots t_m \quad (t_1, t_2, \dots, t_m = 0; 1)$$

closed sets

$$F_n^{t_1}, \ F_n^{t_1 t_2}, \dots, F_n^{t_1 \dots t_m} \quad (n = 1, 2, \dots)$$

have been constructed, with $F_n^0 \subseteq F_n^{t_1 \dots t_m}$ and

$$\Phi\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, F_{q_1+1}^{t_1 t_2}, \dots, F_{q_2}^{t_1 t_2}, \dots; F_{q_{m-2}+1}^{t_1 \dots t_{m-1}}, \dots, F_{q_{m-1}}^{t_1 \dots t_{m-1}}; F_{q_{m-1}+1}^{t_1 \dots t_m}, \dots, F_n^{t_1 \dots t_m}, \dots\} \subset T;$$

3) the points

$x_{t_1 \dots t_m} = \nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}; F_{q_1+1}^{t_1 t_2}, \dots, F_{q_2}^{t_1 t_2}, \dots; F_{q_{m-2}+1}^{t_1 \dots t_{m-1}}, \dots, F_{q_{m-1}}^{t_1 \dots t_{m-1}}; F_{q_{m-1}+1}^{t_1 \dots t_m}, \dots, F_n^{t_1 \dots t_m}, \dots\}$

are distinct for distinct tuples $t_1 t_2 \dots t_m$; 4) the points $x_{t_1 t_2 \dots t_m}$ are surrounded by ε-neighborhoods $S_{t_1 t_2 \dots t_m}$, where $\varepsilon < (1/2)^n$, and the closures of these neighborhoods are pairwise disjoint.

From the continuity of ν on $\Pi t(Z)$ there follows the existence of a number $q_m,\ q_m > q_{m-1}$, such that

$$\nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, \dots; F_{q_{m-1}+1}^{t_1 \dots t_m}, \dots, F_{q_m}^{t_1 \dots t_m}; Q_1, Q_2, \dots, Q_n, \dots\} \in S_{t_1 t_2 \dots t_m},$$

whatever the point $\{Q_n\} \in \Pi t(Z)$.

Introduce the notation:

$$F_n^{t_1 \dots t_m 0} = F_n^{t_1 \dots t_m - 10};$$

$$\nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, \dots; F_{q_{m-1}+1}^{t_1 \dots t_m}, \dots, F_{q_m}^{t_1 \dots t_m}; F_{q_m+1}^{t_1 \dots t_m 0}, \dots, F_n^{t_1 \dots t_m 0}, \dots\} = x_{t_1 \dots t_m 0};$$

$$F_n^{t_1 \dots t_m 1} = F_n^{t_1 \dots t_m 0} + \langle x_{t_1 \dots t_m 0} \rangle;$$

$$\nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, \dots; F_{q_{m-1}+1}^{t_1 \dots t_m}, \dots, F_{q_m}^{t_1 \dots t_m}; F_{q_m+1}^{t_1 \dots t_m 1}, \dots, F_n^{t_1 \dots t_m 1}, \dots\} = x_{t_1 \dots t_m 1}.$$

It is easy to see that

$$x_{t_1 \dots t_m t_{m+1}} \in T \cdot S_{t_1 \dots t_m t_{m+1}}, \quad F_n^0 \subseteq F_n^{t_1 \dots t_m + 1};$$

$x_{t_1 \dots t_m 1} \in \overline{E}$, while

$$x_{t_1 \dots t_m 0} \in \Phi\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, \dots; F_{q_{m-1}+1}^{t_1 \dots t_m}, \dots, F_{q_m}^{t_1 \dots t_m}; F_{q_m+1}^{t_1 \dots t_m 1}, \dots, F_n^{t_1 \dots t_m 1}, \dots\};$$

$x_{t_1 \dots t_m 0} \neq x_{t_1 \dots t_m 1}$.

We construct such ε-neighborhoods $S_{t_1 \dots t_m 0}$ and $S_{t_1 \dots t_m 1}$, $\varepsilon < (1/2)^{m+1}$, respectively of the points $x_{t_1 \dots t_m 0}$ and $x_{t_1 \dots t_m 1}$, that

$$S_{t_1 \dots t_m t_{m+1}} \subset S_{t_1 \dots t_m}, \quad \overline{S}_{t_1 \dots t_m 0} \cdot \overline{S}_{t_1 \dots t_m 1} = 0.$$

The inductively described process gives us, for any k, points $x_{t_1 \dots t_k}$, their ε-neighborhoods $S_{t_1 \dots t_k}$, and closed sets $F_n^{t_1 \dots t_k}$. Their properties show that for any sequence $t_1, t_2, \dots, t_k, \dots$

$$\prod_{k=1}^{\infty} \overline{S}_{t_1 \dots t_k} = \nu\{F_1^{t_1}, \dots, F_{q_1}^{t_1}, \dots; F_{q_{k-1}+1}^{t_1 \dots t_k}, \dots, F_{q_k}^{t_1 \dots t_k}; F_{q_k+1}^{t_1 \dots t_{k+1}}, \dots, F_{q_{k+1}}^{t_1 \dots t_{k+1}}, \dots\}$$

and that, consequently, the set

$$D = \sum_{t_1 t_2 \dots t_k \dots} \prod_{k=1}^{\infty} \overline{S}_{t_1 \dots t_k}$$

is a discontinuum, with $D \subset T \cdot CM$. The theorem is proved.

2°. Here, by Φ will be meant a δs-operation all of whose chains are infinite and which has the following property (ω): whatever sequences of sets $\{Q_{m1}\},\ \{Q_{m2}\}, \dots, \{Q_{mk}\}, \dots$ may be, one has

$$\Phi\{\Phi_1\{Q_{m1}\},\ \Phi_1\{Q_{m2}\}, \dots, \Phi_1\{Q_{mk}\}, \dots\} = \Phi\{Q_1, Q_2, \dots, Q_n, \dots\},$$

where the sequence $\{Q_1, Q_2, \dots, Q_n, \dots\}$ is obtained as a result

arrangement of the sequences of sets $\{Q_{m1}\}, \{Q_{m2}\}, \dots, \{Q_{mk}\}, \dots$ into one sequence of the same sets with the aid of some one-to-one mapping $n = \varphi(mk)$ of the collection of all pairs (mk) of natural numbers onto the natural series $\{n\}$. By Φ_1 is denoted the operation of countable summation. We shall use the following notation:

$$\Phi\{\{Q_{m1}\}, \{Q_{m2}\}, \dots, \{Q_{mk}\}, \dots\} = \Phi\{\Phi_1\{Q_{m1}\}, \Phi_1\{Q_{m2}\}, \dots, \Phi_1\{Q_{mk}\}, \dots\};$$

if $Q_{mk} \in F(Z)$, then

$$\nu\{\{Q_{m1}\}, \dots, \{Q_m\}, \dots\} = \nu\{Q_1, Q_2, \dots, Q_n, \dots\}.$$

R will denote in this subsection a metric space in which the difference of two closed sets is an F_σ-set (for example, R is a space with a countable base). It follows from this that, subtracting from the F_σ-set $\Phi_1\{F_n^0\}$ of the space R a closed set F, we obtain an F_σ-set:

$$\Phi_1\{F_n^0\} - F = \Phi_1\{F_n^1\}.$$

Theorem 2. *If a set T of the space R is effectively distinct from all Φ-sets of this space, then every Φ-set N containing T contains within itself a discontinuum D which does not intersect T: $D \subset N \cdot (R - T)$.*

Let $T \cdot Z = Y$, $N \cdot Z = N_0 = \Phi\{F_n^0\}$, where $\{F_n^0\} \in \Pi_t(Z)$. The set Y is effectively distinct from all Φ-sets of the space Z ((3), p. 136). With the aid of the mapping $\varphi(mk) = n$, represent $\Phi\{F_1^0, \dots, F_n^0, \dots\}$ in the form

$$\Phi\{\{F_{m1}^0\}, \dots, \{F_{mk}^0\}, \dots\};$$

introduce the notation:

$$\nu\{\{F_{m1}^0\}, \dots, \{F_{mk}^0\}, \dots\} = x_0,$$

$$\Phi_1\{F_{mk}^0\} - \langle x_0 \rangle = \Phi_1\{F_{mk}^1\}; \qquad \nu\{\{F_{m1}^1\}, \dots, \{F_{mk}^1\}, \dots\} = x_1.$$

The further construction of the proof presents no difficulty and is carried out analogously to paragraph 1°. The difference in the construction of the points $x_{t_1 \dots t_p}$ consists only in the fact that, instead of the equality

$$F_n^{t_1 \dots t_p 1} = F_n^{t_1 \dots t_p 0} + \langle x_{t_1 \dots t_p 0} \rangle \quad (p.1^\circ)$$

there is the equality

$$\Phi_1\{F_{mk}^{t_1\ldots t_p 1}\} = \Phi_1\{F_{mk}^{t_1\ldots t_p 0}\} - \langle x_{t_1\ldots t_p 0}\rangle.$$

At the end of the construction we arrive at the desired discontinuum

$$D = \sum_{t_1, t_2 \ldots t_p \ldots}^{\infty} \prod_{p=1}^{\infty} \overline{S}_{t_1\ldots t_p},$$

all of whose points are points of the form

$$\nu\{\{F_{m1}^{t_1}\}, \ldots, \{F_{mq_1}^{t_1}\}; \{F_{mq_1+1}^{t_1 t_2}\}, \ldots, \{F_{mq_2}^{t_1 t_2}\}; \ldots; \{F_{mq_{p-1}+1}^{t_1\ldots t_p}\}, \ldots, \{F_{mq_p}^{t_1\ldots t_p}\}; \ldots\}.$$

3°. Considering that Φ is an arbitrary δs-operation, suppose that T ($T \subset R$) is effectively distinct from all Φ-sets of the space R. Let K be the image of the set $\Pi_t(Z)$ under the mapping ν, and suppose that it coincides with the image of the whole set $\Pi_\Phi(R)$ under the same mapping.

It can be proved that the set $T \cdot K + L$, where L is an arbitrary set of the space R not intersecting K, is effectively distinct from all Φ-sets of the space R. From this follows the following proposition on the general type of properties generated by effective distinction. Denote by T a set effectively distinct from all Φ-sets, by Π a universal proposition, and by Σ an existence proposition.

Theorem 3. *In order that the property α be inherent in every set effectively distinct from all Φ-sets, it is necessary and sufficient that the condition*

$$\prod_T \sum_{K \subseteq R} \prod_{L \subseteq R-K} (T \cdot K + L \text{ has property } \alpha)$$

be fulfilled. If R is a Euclidean space, then K is compact.

It follows from this theorem that measurability and Baire' s property do not belong to the general type of properties generated by effective distinction.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.