MATHEMATICS

1958

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Abstract

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MATHEMATICS

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ON THE EXISTENCE OF A SOLUTION OF A PROBLEM OF F. I. FRANKL

(Presented by Academician M. A. Lavrent' ev, 14 X 1957)

Consider the equation

$$\operatorname{sgn} y \cdot |y|^m \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad m > 0, \tag{1}$$

and let D be the domain bounded by: the segment $A'A$ of the axis $x = 0$, $-1 \leq y \leq 1$; a smooth curve Γ, lying in the upper half-plane with endpoints at the points $A(0,1)$ and $B(a,0)$, $a > \frac{2}{m+2}$; the characteristic $A'C$ of equation (1), $A'(0,-1)$, $C\left(\frac{2}{m+2}, 0\right)$, and the segment CB of the axis $y = 0$, $\frac{2}{m+2} \leq x \leq a$.

F. I. Frankl' s problem (1). Find a solution of equation (1), continuous in the closed domain $\overline{D}$ and having continuous partial derivatives u_x and u_y in the closed domain $\overline{D}$ everywhere except, possibly, at the points $O(0,0)$, $A(0,1)$, $A'(0,-1)$, $C\left(\frac{2}{m+2}, 0\right)$, $B(a,0)$, at which they may become infinite of order less than one. In addition, the solution of equation (1) must satisfy the boundary conditions:

$$u\big|_{\Gamma} = \psi_1(s); \tag{2}$$

$$u\big|_{CB} = \psi_2(x); \tag{3}$$

$$\frac{\partial u}{\partial x}\bigg|_{AA'} = 0; \tag{4}$$

$$u(0,y) = u(0,-y), \qquad -1 \leq y \leq 1. \tag{5}$$

This problem was completely solved by A. V. Bitsadze (2) for M. A. Lavrent' ev' s equation. The uniqueness of the solution of F. I. Frankl' s problem for the equation $k(y)u_{xx} + u_{yy} = 0,\ k(0) = 0,\ k(-y) = -k(y),\ k'(y) > 0$, was considered by A. V. Bitsadze ([3]).

By means of a method analogous to that used in ([4]), in note ([5]) the uniqueness of the solution of this problem was proved under somewhat different restrictions imposed on the contour Γ, and, for $a = \dfrac{2}{m+2}$, the existence of a solution was proved.

Without loss of generality, one may assume that $\psi_1 = \psi_2 \equiv 0$. With regard to the curve Γ, we shall suppose that it satisfies the conditions of the uniqueness theorem for the solution of F. I. Frankl' s problem; the coordinates of its points as functions of the arc s have continuous second derivatives for $0 \le s \le l$, where l is the length of the contour Γ, and in a neighborhood of the point B on the curve Γ the condition $|dx/ds| \le c_1 y^{m/2}(s)$ is fulfilled.

Denote by D^* the domain symmetric to the domain D with respect to the axis $x = 0$, and let $D_1 = D + D^* + A'A$. Let Γ^* be the contour symmetric to Γ, and let the points B^* and C^* be symmetric, respectively, to the points B and C with respect to the axis $x = 0$.

To solve the problem of F. I. Frankl, let us solve the auxiliary problem: find a solution of equation (1) with boundary conditions

$$u\,\Big|_{\Gamma+\Gamma^*} = 0; \tag{6}$$

$$u\,\Big|_{CB+B^*C^*} = 0 \tag{7}$$

and satisfying condition (5), where the functions $\tau(x) = u(x,0)$ and $\nu(x) = \dfrac{\partial u}{\partial y}(x,0)$, $-\dfrac{2}{m+2} \le x \le \dfrac{2}{m+2}$, will be assumed even.

In this case condition (4) is always fulfilled, and for $x \ge 0$ the solution of this auxiliary problem will be a solution of the problem of F. I. Frankl.

The solution of equation (1) in the hyperbolic part D_1 of the domain, satisfying the conditions $u(x,0) = \tau(x)$ and $\dfrac{\partial u}{\partial y}(x,0) = \nu(x)$, has the form:

$$\begin{aligned} u(x,y) = \gamma_2 y \int_0^1 \nu\big[x + (1-2\beta)(-y)^{(m+2)/2}(2t-1)\big]\, t^{-\beta}(1-t)^{-\beta}\, dt \\ + \gamma_1 \int_0^1 \tau\big[x + (1-2\beta)(-y)^{(m+2)/2}(2t-1)\big]\, t^{\beta-1}(1-t)^{\beta-1}\, dt, \end{aligned} \tag{8}$$

where

$$\gamma_1 = \Gamma(2\beta)/\Gamma^2(\beta), \qquad \gamma_2 = \Gamma(2-2\beta)/\Gamma^2(1-\beta), \qquad \beta = m/2(m+2).$$

Using condition (5), from this equality one can obtain a relation connecting the functions $\tau(x)$, $\nu(x)$, and $\varphi(y) = u(0,y)$, $0 \le y \le 1$:

$$2^{1-2\beta}\frac{\Gamma(1-\beta)\Gamma(2\beta)}{\Gamma(\beta)}\tau(x) = (1-2\beta)^{2\beta}\int_0^x \frac{\nu(t)\,dt}{(x-t)^{2\beta}} + \frac{d}{dx}\int_0^x \frac{y^{2\beta}\varphi\left[\left(\frac{y}{1-2\beta}\right)^{1-2\beta}\right]}{(x^2-y^2)^{\beta}}\,dy - \frac{d}{dx}\int_0^x \frac{y^{2\beta}f\left[\left(\frac{y}{1-2\beta}\right)^{1-2\beta}\right]}{(x^2-y^2)^{\beta}} \tag{9}$$

$$0 \le x \le 1-2\beta.$$

From this relation we shall eliminate the functions $\varphi(x)$ and $\nu(x)$, expressing them in terms of $\tau(x)$. To do this we use the solution of the Dirichlet problem for equation (1) in the elliptic part D_1^+ of the domain D_1 with boundary conditions (6), (7) and $u(x,0) = \tau(x)$ for $-\dfrac{2}{m+2} \le x \le \dfrac{2}{m+2}$. The Dirichlet problem for the domain D_1^+ is solved with the aid of the Green function, which is constructed in the same way as in Gellerstedt ([6]). If $G_0(x,y;x_0,y_0)$ denotes the Green function for the domain bounded by the "normal"curve $\Gamma_0:\ x^2+(1-2\beta)^2y^{m+2} = a^2$ and the segment of the axis $y=0$, $-a \le x \le a$ ([7]), then the Green function for the domain D_1^+ can be represented in the form

$$G(x,y;x_0,y_0) = G_0(x,y;x_0,y_0) + H(x,y;x_0,y_0).$$

The solution of the Dirichlet problem for the domain D_1^+, vanishing on $\Gamma + \Gamma^*$ and on the segments B^*C^* and CB of the axis $y=0$ and equal to $\tau(x)$ for $-(1-2\beta) \le x \le (1-2\beta)$, has the form:

$$u(x,y) = \int_{-(1-2\beta)}^{1-2\beta} \tau(\xi)\left[\frac{\partial G_0(\xi,0;x,y)}{\partial\eta} + \frac{\partial H(\xi,0;x,y)}{\partial\eta}\right]d\xi. \tag{10}$$

Let us determine from this equality $\nu(x)$ and $\varphi(x)$ and substitute into relation (9). After the necessary simplifications we obtain the integral equation

$$\tau(x) - 4\lambda\int_0^{1-2\beta}\tau(\xi)x^{1-2\beta}\left(\frac{\xi^{2+2\beta}}{\xi^4-x^4} - \frac{a^{2+2\beta}}{a^4 - \dfrac{x^4\xi^4}{a^4}}\right)d\xi -$$

$$-2\lambda\int_0^{1-2\beta}\tau(\xi)H_1(\xi,x)\,d\xi = F(x), \tag{11}$$

where

$$\lambda = \frac{\cos\beta\pi}{\pi(1+\sin\beta\pi)},$$

$$H_1(\xi, x) = \frac{\pi\gamma_2}{2^{1-2\beta}(1-2\beta)}\times$$

$$\times\frac{d}{dx}\int_0^x \frac{y^{2\beta}\,dy}{(x^2-y^2)^\beta}\left[\frac{\partial H}{\partial\eta}\left(\xi, 0; 0, \left(\frac{y}{1-2\beta}\right)^{1-2\beta}\right) + \frac{\partial H}{\partial\eta}\left(-\xi, 0; 0, \left(\frac{y}{1-2\beta}\right)^{1-2\beta}\right)\right]+$$

$$+\frac{\pi\gamma_2}{[2(1-2\beta)]^{1-2\beta}}\int_0^x \frac{dt}{(x-t)^\beta}\left[\frac{\partial^2 H}{\partial\eta\,\partial y}(-\xi, 0; t, 0) + \frac{\partial^2 H}{\partial\eta\,\partial y}(\xi, 0; t, 0)\right],$$

$$F(x) = -\frac{\lambda\pi\gamma_2\cdot 2^{2\beta}}{1-2\beta}\frac{d}{dx}\int_0^x \frac{y^{2\beta} f\left[\left(\frac{y}{1-2\beta}\right)^{1-2\beta}\right]}{(x^2-y^2)^\beta}\,dy.$$

It can be shown that, under our assumptions concerning the contour Γ, the function $H_1(\xi, x)$ is continuous in both variables and bounded. Now we use the solution of the equation

$$\psi(x) - \lambda\int_0^1 \psi(\xi)\left[\frac{1}{\xi - x} - \frac{b}{b^2 - x\xi}\right]d\xi = \Psi(x), \qquad b > 1, \tag{12}$$

which has the form

$$\psi(x) = \frac{\lambda}{1+\lambda^2\pi^2}\frac{(1-x)^\alpha}{x^\alpha}(b^2-x)^\alpha\int_0^1 \frac{t^\alpha\Psi(t)}{(1-t)^\alpha(b^2-t)^\alpha}\left(\frac{1}{t-x} - \frac{b}{b^2-tx}\right)dt+$$

$$+\frac{\Psi(x)}{1+\lambda^2\pi^2} + \frac{c}{x^\alpha(1-x)^{1-\alpha}(b^2-x)^{1-\alpha}},$$

where c is an arbitrary constant,

$$\alpha = \frac{1-2\beta}{4}.$$

If in equation (11) we replace x by $4\alpha\sqrt[4]{x}$ and ξ by $4\alpha\sqrt[4]{\xi}$, then equation (11) takes the form (12), where

$$\psi(x) = x^{-\alpha}\tau(4\alpha\sqrt[4]{x}), \qquad b = \left(\frac{a}{1-2\beta}\right)^4,$$

$$\Psi(x) = x^{-\alpha}F(4\alpha\sqrt[4]{x}) + \lambda\int_0^1 \tau(4\alpha\sqrt[4]{\xi})\,2\alpha x^{-\alpha}\xi^{-3/4}H_1(4\alpha\sqrt[4]{\xi}, 4\alpha\sqrt[4]{x})\,d\xi +$$

$$+\lambda\int_0^1 \tau(4\alpha\sqrt[4]{\xi})\,b\xi^{-\alpha}\frac{1-(b/\xi)^{(1+\beta)/2}}{b^2-x\xi}\,d\xi.$$

Substituting into the solution of equation (12) the corresponding values for $\psi(x)$, b, and $\Psi(x)$, we obtain the Fredholm integral equation with respect to $\tau_1(x) = \tau(4a\sqrt[4]{x})$:

$$\tau_1(x) - \lambda_1\int_0^1 K_1(\xi, x)\tau_1(\xi)\,d\xi = F_1(x), \tag{13}$$

where

$$\lambda_1 = \frac{\lambda^2}{1+\lambda^2\pi^2},$$

$$K_1(\xi, x) = \frac{b(\xi^{(1+\beta)/2} - b^{(1+\beta)/2})}{\xi^{3/4}(b^2-x\xi)}\int_0^1 \left[\frac{(1-x)(b^2-x)}{(1-t)(b^2-t)}\right]^\alpha \frac{t^\alpha}{b^2-t\xi}$$
$$\times\left[-\frac{b(b^2-x\xi)}{b^2-tx} + \frac{(b^2-x\xi) - \frac{x}{t}(b^2-t\xi)\left(\frac{b^2-t}{b^2-x}\right)^\alpha}{t-x}\right]dt$$
$$-\frac{2\alpha}{\xi^{3/4}}\int_0^1 \left[\frac{(1-x)(b^2-x)}{(1-t)(b^2-t)}\right]^\alpha \left[b\,\frac{H_1(4a\sqrt[4]{\xi},\,4a\sqrt[4]{t})}{b^2-tx}\right.$$
$$\left.-\frac{H_1(4a\sqrt[4]{\xi},\,4a\sqrt[4]{t}) - H_1(4a\sqrt[4]{\xi},\,4a\sqrt[4]{x})\left(\frac{b^2-t}{b^2-x}\right)^\alpha\left(\frac{x}{t}\right)^{1-\alpha}}{t-x}\right]dt,$$

$$F_1(x) = \frac{\lambda}{1+\lambda^2\pi^2}\int_0^1 \left[\frac{(1-x)(b^2-x)}{(1-t)(b^2-t)}\right]^\alpha \frac{F(4a\sqrt[4]{t}) - F(4a\sqrt[4]{x})\left(\frac{b^2-t}{b^2-x}\right)^\alpha \left(\frac{x}{t}\right)^{1-\alpha}}{t-x}\,dt$$
$$- \frac{\lambda b}{1+\lambda^2\pi^2}\int_0^1 \left[\frac{(1-x)(b^2-x)}{(1-t)(b^2-t)}\right]^\alpha \frac{F(4a\sqrt[4]{t})}{b^2-tx}\,dt + \frac{c}{(1-x)^{1-\alpha}(b^2-x)^{1-\alpha}}. \quad (14)$$

Equation (13) is equivalent to F. I. Frankl' s problem. Fredholm theory is applicable to this equation, and, consequently, from the uniqueness of the solution of F. I. Frankl' s problem there will follow the solvability of equation (13). Having solved it, we find the function $\tau(x)$, and by formula (10) we obtain the solution of F. I. Frankl' s problem for the elliptic part of the domain D. Then, determining $\nu(x)$, one can also obtain the solution in the hyperbolic part of the domain D, using equality (8). To obtain a solution of equation (13) that is continuous at $x = 1$, it is necessary to put $c = 0$ in (14). In the special case when Γ is a "normal" contour passing through the points A and B, and $a = 1 - 2\beta$, $H_1(\xi, x) \equiv 0$, an equation is obtained similar to that considered in note [5].

In conclusion I consider it my pleasant duty to express my deep gratitude to A. V. Bitsadze for posing the problem, for his constant attention, and for valuable advice during the preparation of the present work.

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Received
21 IX 1957

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Note: Figure translations are in progress. See original paper for figures.