# Mathematics

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## Abstract

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## The Use of Fractional Powers of Operators in the Study of Fourier Series in Eigenfunctions of Differential Operators

*(Presented by Academician S. L. Sobolev on 5 VI 1958)*

In recent years, interest has increased in the study of Fourier series in the eigenfunctions of the Laplace operator and of other differential operators (see, for example, (1–3) and the bibliography cited there). Questions have been studied concerning conditions for expandability into uniformly convergent series, the rate of convergence of these series, the applicability of the Fourier method to the solution of boundary-value problems, termwise differentiation of series, and so on.

Quite simple and transparent answers to a number of the questions listed above are obtained, as it seems to us, by considerations using negative fractional powers of differential operators. All difficulties turn out to be reduced to determining from which functional spaces into which ones the negative powers of operators act, and this, for concrete differential operators, is essentially known. We note that, in a closely related situation, fractional powers of operators were used in the theory of nonlinear integral equations (4), and in the theory of parabolic equations (5).

1. Let $T$ be a positive-definite self-adjoint operator in a Hilbert space $H$, possessing a completely continuous inverse. We denote the eigenvalues and eigenfunctions of the operator $T$ by $\lambda_i$ and $u_i$: $Tu_i = \lambda_i u_i$; by $\Omega_\alpha$ we denote the domains of definition of the operators $T^\alpha$ $(\alpha > 0)$.

**Theorem 1.** *Suppose the operator $T^{-\beta}$ acts from $H$ into some space $E \subset H$ and is a continuous operator. Let $f \in \Omega_{\beta+\gamma}$ $(\gamma \geq 0)$. Then the Fourier series*

$$(f, u_1)u_1 + (f, u_2)u_2 + \cdots + (f, u_n)u_n + \cdots \tag{1}$$

*converges to $f$ in the norm of the space $E$, and the rate of convergence is characterized by the estimate*

$$\left\| f - \sum_{i=1}^{n} (f, u_i)u_i \right\|_E = o\left(a_n^{-\gamma}\right),$$

*where $a_n$ is the least of the numbers $\lambda_{n+1}, \lambda_{n+2}, \ldots$.*

The assertion of the theorem is almost obvious. Indeed, since $f \in \Omega_{\beta+\gamma}$, we have $f = T^{-\beta-\gamma} g$, where $g \in H$. Therefore

$$\left\| f - \sum_{i=1}^{n} (f, u_i) u_i \right\|_E \leq \left\| T^{-\beta} \right\|_{H \to E} a_n^{-\gamma} \left[ \sum_{i=n+1}^{\infty} (g, u_i)^2 \right]^{1/2} .$$

The series (1) converge to $f$ under any permutation of their terms; therefore, in the case where $E$ is the space $C$ of continuous functions, the series (1) converges to $f$ not only uniformly but also absolutely. Let us note that $a_n = \lambda_n + 1$, if the $\lambda_i$ are numbered in their natural increasing order.

2. The strongest consequences on the absolute and uniform convergence of Fourier series in orthonormal eigenfunctions $u_i$ of concrete operators $T$ can be obtained if one knows which smallest negative powers of the operator $T$ act from $L^2$ into $C$. The question of negative integer powers of operators acting from $L^2$ into $C$ is easily solved on the basis of the theorems of S. L. Sobolev ([6]) on operators of potential type, if the singularities of the Green' s function of the operator $T$ are known. For the case of fractional powers, in many cases the answer is given by a theorem which, in its main part, follows from the theorems on operators of potential type and from certain theorems on the splitting of operators ([4]).

**Theorem 2.** Let the symmetric positive-definite kernel $K(P, Q)$ ($P, Q \in G$, $G$ a closed bounded domain of the $N$-dimensional space) satisfy the inequality $|K(P, Q)| \leq M r^{-\lambda}$ ($P, Q \in G$), where $\frac{1}{2} N < \lambda < N$. Let

$$A\varphi(P) = \int_G K(P, Q) \varphi(Q) \, dQ.$$

Let $\alpha_0 = N/2(N-\lambda)$. Then, for $\alpha > \alpha_0$, the operator $A^\alpha$ acts from $L^2$ into the space of bounded functions and is completely continuous (if the eigenfunctions of the kernel are continuous, then $A^\alpha$, for $\alpha > \alpha_0$, acts from $L^2$ into $C$ and is completely continuous). The operator $A^{\alpha_0}$ acts from $L^2$ into any $L^p$ and is completely continuous. For $\alpha < \alpha_0$ the operator $A^\alpha$ acts from $L^2$ into every

$$L^p, \quad \text{where } p < \frac{2N}{N - 2\alpha(N-\lambda)}$$

and is completely continuous.

Apparently, Theorem 2 is also true for $\lambda > \frac{1}{2} N$. For a number of concrete cases this can be proved quite simply, if one combines the results of V. A. Il' in on fractional powers of the Laplace operator with the remarkable inequalities of Heinz on fractional powers of operators ([11]).

We note that some theorems of V. A. Il' in on negative fractional powers of the Laplace operator are, in turn, consequences of Theorem 2.

If the singularities of the Green's function of the operator $T$ are known, then the question of conditions for the uniform and absolute convergence of Fourier series is solved by combining Theorems 1 and 2. The conditions of expandability are then formulated in terms of the membership of $f$ in the domain of definition of a certain power of the operator $T$. To obtain from the considerations set forth the theorems of O. A. Ladyzhenskaya ([1]) and certain stronger theorems of V. A. Il' in, it is enough to take into account that the domain of definition of the square root of the Laplace operator coincides with $W_2^1$ in the case of the second and third boundary-value problems, and with the set of those functions in $W_2^1$ that satisfy the boundary condition in the case of the first boundary-value problem.

3. **Theorem 3.** Let a self-adjoint positive-definite operator $A$ in $L^2$ act from $L^2$ into $C$ and be continuous. Then the operator $A^\alpha$, $0 < \alpha < 1$, acts from $L^2$ into $L^p$, where

$$p \leq \frac{2}{1-\alpha}.$$

If, moreover, the operator $A$ is completely continuous, then $A^\alpha$, as an operator acting from $L^2$ into $L^p$, is also completely continuous.

Let $A$ act from $L^2$ into $L^{p_0}$, where $p_0 > 2$, and be continuous. Then the operator $A^\alpha$, $0 < \alpha < 1$, acts from $L^2$ into $L^p$, where $p \leq p_1$ and

$$p_1 = \frac{2p_0}{(1-\alpha)p_0 + 2\alpha},$$

and moreover

$$\|A^\alpha\|_{L^2 \to L^{p_1}} \leq \left(\|A\|_{L^2 \to L^{p_0}}\right)^\alpha.$$

If, moreover, the ope-

operator $A$ is completely continuous, then the operators $A^\alpha$, acting from $L^2$ into $L^p$, are also completely continuous.

As applied to abstract operators, Theorem 3 is a strengthening of results obtained earlier by one of the authors ([4]). A relatively simple proof of Theorem 3 was obtained on the basis of the well-known theorem of M. Riesz on the logarithmic convexity of the norm of operators acting in the spaces $L^p$, and of certain estimates from ([7]). The above-mentioned Riesz theorem makes it possible to infer from Theorem 3 where the operators $A^\alpha$, continued by continuity, act on certain spaces $L^q$, where $q < 2$.

4. The problem of termwise differentiation of Fourier series is also solved simply (without estimates for derivatives of Green' s functions), if one uses the notion of subordination of one operator to a fractional power of another. The question of the conditions of subordination, as was clarified by S. G. Krein, P. E. Sobolevskii, and one of the authors, plays an important role in the theory of linear and nonlinear parabolic equations.

Below, by $D^m$ we denote differentiation operators of order $m$ (in the sense of S. L. Sobolev).

**Theorem 4.** Let some operator $D^k$ be subordinate to the operator $T^\beta$ in the sense that

$$\left\|D^k T^{-\beta}\varphi\right\|_E \leq b\|\varphi\|_H \qquad (\varphi \in H). \tag{2}$$

Let $f \in \Omega_{\beta+\gamma}$. Then the operator $D^k$ may be applied termwise to the series (1) (the series may be termwise differentiated $k$ times), and the resulting series will converge in the norm of $E$; the rate of convergence is characterized by the inequality

$$\left\|D^k f - \sum_{i=1}^{n}(f, u_i)D^k u_i\right\|_E = o(a_n^{-\gamma}).$$

If $E$ is the space $C$ of continuous functions, then Theorem 4 gives conditions for uniform and absolute convergence (in the whole closed domain!) of termwise differentiated Fourier series.

By virtue of the embedding theorems of S. L. Sobolev, in order for condition (2) to hold, where $E$ is $C$, it is sufficient that the inequalities

$$\left\|D^{\left[\frac{N+1}{2}\right]+k}T^{-\beta}\right\|_{L^2} \leq b\|\varphi\|_{L^2}.$$

hold. As applied to differential operators $T$, the question of subordination conditions of the form (2) has recently been solved practically completely by V. P. Glushko and S. G. Krein ([10]), who proceeded from the conditions of subordination of abstract operators established earlier by S. G. Krein and P. E. Sobolevskii in ([8]).

The conditions from ([8]) are conveniently formulated in the following form.

**Theorem 5.** Let $T$ be a self-adjoint positive-definite operator. In order that the operator $T_1$, admitting a closure, be subordinate to all operators $T^\beta$, $\beta > \beta_0 \geq 0$ ($\|T_1\varphi\| \leq K(\beta)\|T^\beta\varphi\|$ for $\varphi \in \Omega_\beta$), it is necessary and sufficient that the inequality

$$\|T_1\varphi\|_H \leq K_1(\beta)\|T\varphi\|^\beta \cdot \|\varphi\|^{1-\beta} \qquad (\varphi \in \Omega_\beta,\ \beta > \beta_0) \tag{3}$$

hold.

The necessity of condition (3) follows from the simple inequality

$$\|T^{\beta}\varphi\|_H \leq \|T\varphi\|_H^{\beta} \cdot \|\varphi\|_H^{1-\beta},$$

established earlier by one of the authors in connection with the problem of decomposing a linear operator ([4]).

5. The considerations set forth are applicable in justifying the Fourier method for solving parabolic and hyperbolic equations. Consider, for example, the problem

$$\frac{d^2x(t)}{dt^2} + Tx(t) = f(t), \qquad x(0) = \varphi_0, \qquad x'_t(0) = \psi_0, \tag{4}$$

where $T$, as above, is a positive-definite operator, having a completely continuous inverse, acting in $H$. The solution of problem (4) can be written in the form

$$x(t) = \cos(T^{1/2}t)\varphi_0 + T^{-1/2}\sin(T^{1/2}t)\psi_0 + T^{-1/2}\int_0^t \sin(T^{1/2}t - T^{1/2}\tau)f(\tau)\,d\tau. \tag{5}$$

The question of the convergence of the Fourier method coincides with the question of the convergence of the ordinary Fourier expansions of the functions (5). To study these series (and the series obtained by termwise differentiation with respect to the spatial coordinates or with respect to $t$), the arguments that led to Theorems 1 and 4 are applicable. If, for example, the operator $T^{-\alpha}$ acts from $H$ into $C$, then for uniform and absolute convergence of the Fourier method it is sufficient that the values of the function (5) belong to $\Omega_\alpha$; this will be fulfilled if $\varphi_0 \in \Omega_\alpha$, $\psi_0 \in \Omega_{\alpha-1/2}$, $f(t) \in \Omega_{\alpha-1/2}$. Conditions for the convergence of series obtained by termwise differentiation are formulated analogously, and estimates of the rate of convergence are indicated. For lack of space we do not give the exact formulations of the corresponding theorems.

The scheme described immediately leads to the theorems of O. A. Ladyzhenskaya and V. A. Il' in on the convergence of the Fourier method for hyperbolic equations with the Laplace operator. Analogous theorems are also established for equations with other differential operators.

6. A more complicated question than the absolute convergence of a Fourier series is the question of uniform conditional convergence. Subtle theorems in this direction were obtained by B. M. Levitan ([9]) and V. A. Il' in ([2]). It has not yet been possible to obtain analogous theorems by operator considerations. This could be done if two difficult problems were solved.

First, it is necessary to know what minimal power $T^{-\alpha}$ of the operator $T$ acts from $L^p$, where $p > 2$, into the space $C$. Second, it is necessary to know under what minimal restrictions the series (1) converges to $f$ in the norm of the space $L^p$, where $p > 2$. Combining these conditions will yield theorems on the uniform convergence of the Fourier series of the function $T^{-\alpha} f$ (convergence to $f$ of series in $L^p$ may be destroyed under rearrangement—hence it follows that for the Fourier series of the function $T^{-\alpha} f$ one can guarantee only conditional convergence!).

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*Note: Figure translations are in progress. See original paper for figures.*