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Abstract

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ON THE RATE OF CONVERGENCE OF AN ITERATIVE PROCESS FOR THE FINITE-DIFFERENCE SOLUTION OF THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION

(Presented by Academician S. L. Sobolev on 27 XI 1957)

1. In the paper ([1]) a family of operators D_α, associated with the operator

$$Du_{ij} = {}^1/_4(u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j})$$

by the relation

$$D_\alpha u = \frac{1}{1+\alpha}(Du + \alpha u),$$

was introduced.

The operator corresponding to $\alpha = {}^1/_4$ will have the form*

$$D_{1/4} = {}^4/_5(D + 1),$$

and its application reduces to replacing the value of the function at each node of the grid by the arithmetic mean of the values at 5 nodes—the given one and the 4 neighboring ones:

$$D_{1/4}u_{i,j} = {}^1/_5(u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j} + u_{i,j}).$$

Since this operator is only slightly more complicated than D, and division by 5 reduces to doubling the number and division by 10, i.e., to shifting the decimal point, it is of interest to determine the advisability of applying the operator $D_{1/4}$. The relevance of posing such a question is confirmed by the fact that in the majority of general-purpose electronic digital machines currently being built, the decimal number system is used, which is apparently explained by the broad possibility of self-checking provided by the binary-coded decimal system and by the absence of the need to convert numbers from decimal form into binary and back. For such machines the operation D consists of 3 short and one long command, while the operation $D_{1/4}$ consists of 6 short ones. Hence one cannot conclude that for machines for which $\tau_2 > 3\tau_1$ (τ_1 is the time required to execute a short command—addition, shift; τ_2 is the time required to execute a long command—division, multiplication), it is more expedient to use the operator $D_{1/4}$.

The eigenvalues K_α of the operator D_α are related to the eigenvalues K of the operator D by the relation

$$K_\alpha = \frac{K+\alpha}{1+\alpha}.$$

Hence, in particular, it follows that for $\alpha \geqslant 0$ the operator D_α yields a convergent process. The number of iterations for solving the finite-difference analogue of the Dirichlet problem for the Laplace equation in a square of side N with accuracy 10^{-k} is approximately equal to the least integer $m \geqslant$

$$\left|\frac{\log\left(\dfrac{10^{-k}\cdot 8h^2}{N^2}\right)}{\log K^*}\right|,$$

where

* In paper ([1]) on p. 57 there is a misprint: everywhere instead of $\alpha = {}^1/_5$ it should be $\alpha = {}^1/_4$.

K^* is the spectral norm (maximum modulus) of the eigenvalues of the operator used, h is the grid step. The total computation time is

$$T_1 = \left|\frac{\log\left(\dfrac{10^{-k}\cdot 8h^2}{N^2}\right)}{\log K^*}\right|(N-1)^2(L_1\tau_1 + L_2\tau_2),$$

where L_1 is the number of short operations, and L_2 is the number of long operations.

Analogously, for the operator D_α one can write

$$T_2 = \left|\frac{\log\left(\dfrac{10^{-k}\cdot 8h^2}{N^2}\right)}{\log K_\alpha^*}\right|(N-1)^2(L_3\tau_1 + L_4\tau_2).$$

For the ratio of times we have

$$\frac{T_1}{T_2} = \left|\frac{\log K_\alpha^*}{\log K^*}\right| \frac{L_1\tau_1 + L_2\tau_2}{L_3\tau_1 + L_4\tau_2}.$$

Let us compute $\log K_\alpha^* / \log K^*$ for $\alpha = 1/4$ in the case of Richardson iteration for large N:

$$\frac{\ln K_{1/4}^*}{\ln K^*} = \frac{\ln\left(\frac{4K^*+1}{5}\right)}{\ln K^*} = \frac{\ln\left(1 - \frac{\pi^2 N^{-2}}{10}\right)}{\ln\left(1 - \frac{\pi^2 N^{-2}}{8}\right)} \approx= 0.8.$$

It is easy to show that this value is also valid for the Liebmann iterative process. For $T_1 = T_2$ we obtain

$$\frac{\tau_1}{\tau_2} = \frac{0.8L_2 - L_4}{L_3 - 0.8L_1}.$$

In automating the solution of the Dirichlet problem (6), $L_1 = 24,\ L_2 = 1,\ L_3 = 27,\ L_4 = 0$, and $\tau_1/\tau_2 = 0.1$. In manual computation $L_1 = 3,\ L_2 = 1,\ L_3 = 6,\ L_4 = 0$, and $\tau_1/\tau_2 = 0.2$, i.e., division by 4 is replaced by four additions and one transfer of the decimal point.

2. It is clear that, without investigating the spectrum of the eigenvalues of D, one cannot use negative α, since it may happen that the spectral norm of the operator D_α becomes greater than unity. Therefore, in (1) values $\alpha < 0$ are not considered. We shall show that for the Liebmann iterative process, when $\alpha = -1/4$, one obtains a very compact iterative scheme and a considerable gain in the rate of convergence.

In (3) it is proved that the eigenvalues of the operator D for a rectangular domain and the Liebmann iterative process are equal to

$$K = \frac{1}{4}\left[\cos\left(\frac{\pi r}{q}\right) + \cos\left(\frac{\pi s}{p}\right)\right]^2 \quad (r = 1, 2, \dots, q-1;\ s = 1, 2, \dots, p-1).$$

One of the largest eigenvalues is obtained for $r = s = 1$. For large p and q,

$$K^* = \left[1 - \frac{\pi^2(p^{-2} + q^{-2})}{4}\right]^2.$$

Since all eigenvalues are positive, negative values are admissible for α. The positivity of all eigenvalues of the Liebmann iterative process gave rise to the search for methods of accelerating convergence and led to more advanced iterative methods (the extrapolated Liebmann method ([3]); the successive over-relaxation method ([4]), coinciding with the extrapolated Liebmann method in the case of the Laplace equation; and the group over-relaxation method ([5])). For an arbitrary domain Q contained inside the domain Q_1, it is proved in ([2]) that the spectral norm $K^*(S, Q)$ of any operator approximately replacing the Laplace operator is less than or equal to $K^*(S, Q_1)$.

For $\alpha = -1/4$ one obtains the operator

$$D_{-1/4} u_{i,j} = \frac{1}{3} \left(u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j} - u_{i,j} \right).$$

The spectral norm of this operator in the case of Liebmann iteration is equal to

$$K^*_{-1/4} = \frac{\left[4 - \pi^2 \left(p^{-2} + q^{-2}\right)\right]^2 - 4}{12}.$$

Let us write the computation time, without distinguishing short and long operations:

$$T = \left| \frac{\log \left(\frac{8 \cdot 10^{-k}}{N^2} \right)}{\log K^*} \right| (N-1)^2 L\tau.$$

For large N we obtain

$$T_{\text{Rich}} = 2S(N,k)L_1\tau, \qquad T_{\text{Lieb}} = S(N,k)L_1\tau,$$

$$T_{\alpha=-1/4} = \frac{3}{4} S(N,k)(L_1+1)\tau,$$

where

$$S(N,k) = \ln \left(\frac{8 \cdot 10^{-k}}{N^2} \right) \frac{N^2}{\pi^2} (N-1)^2.$$

It follows immediately from this that Liebmann iteration converges twice as fast as Richardson iteration.

Table 1 gives the results for $k = 6$.

Table 1

N	T_{Rich}, $L_1 = 4$	T_{Rich}, $L_1 = 24$	T_{Lieb}, $L_1 = 4$	T_{Lieb}, $L_1 = 24$	$T_{\alpha=-1/4}$, $L_1 = 4$	$T_{\alpha=-1/4}$, $L_1 = 24$
10	$9 \cdot 10^4\tau$	$3.6 \cdot 10^5\tau$	$4.5 \cdot 10^4\tau$	$1.8 \cdot 10^5\tau$	$4.2 \cdot 10^4\tau$	$1.3 \cdot 10^5\tau$
20	$2 \cdot 10^6\tau$	$8 \cdot 10^6\tau$	$10^6\tau$	$4 \cdot 10^6\tau$	$9.4 \cdot 10^5\tau$	$3 \cdot 10^6\tau$
30	$1.1 \cdot 10^7\tau$	$4.4 \cdot 10^7\tau$	$6 \cdot 10^6\tau$	$2.2 \cdot 10^7\tau$	$5.4 \cdot 10^6\tau$	$1.7 \cdot 10^7\tau$

As is seen, the operator $D_{-1/4}$ requires a smaller number of arithmetic operations than the operator D. With automation ($L_1 = 24$) the saving in time is considerable.

It is easy to verify that for $K^* = 1/3$ the operator $D_{-1/4}$ will have the same convergence as D, and for a smaller value of K^* even somewhat slower. In automating the solution of the Dirichlet boundary-value problem on BESM for $K^* = 1/3$, the computation time is about 16 sec.; therefore the decrease of convergence in this region of K^* may be neglected.

A further decrease of α to $-1/2$ increases the value of K at which D_α gives poorer convergence; for $\alpha = -1/2$, $K^*_\alpha = -1$, independently of the dimensions of the domain, while for $\alpha < -3/4$ one already obtains operators leading to an unstable scheme for solving the Dirichlet problem.

Table 2

	$N = 10$	$N = 20$	For large N
Laplace–Richardson	$1.2 \cdot 10^5\tau$	$2.2 \cdot 10^6\tau$	$2cN^4\tau^*$
Laplace–Liebmann	$6 \cdot 10^4\tau$	$1.1 \cdot 10^6\tau$	$cN^4\tau$
Laplace–Liebmann (extrapol.)	$1.6 \cdot 10^4\tau$	$1.5 \cdot 10^5\tau$	$2.5cN^3\tau$
Laplace–Richardson (2nd order)	$4 \cdot 10^4\tau$	$4 \cdot 10^5\tau$	$7cN^3\tau$
Biharmonic–Richardson	$1.4 \cdot 10^7\tau$	$1.1 \cdot 10^8\tau$	$2.5cN^6\tau$
Biharmonic–Richardson (2nd order)	$5 \cdot 10^5\tau$	$10^7\tau$	$8cN^4\tau$

$$^*\, c = \left| \ln \frac{2 \cdot 10^{-6}}{(N+1)^2} \right|.$$

For the optimal α we obtain

$$\alpha_{\text{opt}} = -\frac{\frac{K^* - K_{\min}}{2}}{K^*} \simeq -\frac{K^*}{2},$$

whence

$$K^*_{\alpha_{\text{opt}}} = \frac{K^*}{2 - K^*}.$$

Write K^* in the form

$$K^* = \frac{\frac{K^*}{1-K^*}}{\frac{K^*}{1-K^*} + 1}.$$

Then

$$K_{\alpha_{\text{opt}}} = -\frac{\frac{K^*}{1-K^*}}{\frac{K^*}{1-K^*} + 2},$$

and, as $K \to 1$, the difference $(K^* - K^*_{\alpha_{\text{opt}}}) \to 0$ on the right as $\dfrac{K-1}{K}$.

It can be shown that for the extrapolated Liebmann method

$$K_1^* = \frac{2 - K^* - 2\sqrt{1-K^*}}{K^*}.$$

The difference

$$K^*_{\alpha_{\text{opt}}} - K_1^* = \frac{K^*}{2-K^*} - \frac{2 - K^* - 2\sqrt{1-K^*}}{K^*} > 0$$

for K^* in the interval $(0, 1)$. Therefore the α optimal in the sense of the extrapolated Liebmann method gives a greater saving of time than the α optimal in the sense of the iterator D_α.

In works ([3],[4]), as the number of operations necessary for solving the difference analogue of the Dirichlet problem with accuracy 10^{-k}, the least integer is taken:

$$m \geq \left[\frac{\log(10^{-k})}{\log K^*}\right]$$

(it is assumed that $|u(M) - u_0(M)| \leq 1$). It is more convenient to relate the number of iterations to the residuals. Then, assuming that $|u_1(M) - u_0(M)| \leq 1$, after m iterations the residuals will be less than 10^{-k}, but the accuracy of the solution of the difference equation will in this case be, by the well-known Gershgorin formula, approximately equal to $10^{-k} l^2/4h^2$, where l is the radius of a circle containing the given domain entirely.

Table 2 gives the recomputed estimates ([3]) of the time for solving the equation Δu and $\Delta\Delta u$ with accuracy 10^{-6}.

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Note: Figure translations are in progress. See original paper for figures.