SovietRxiv

Soviet-era science, translated into English

MATHEMATICS

Academician A. N. KOLMOGOROV

1958

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-195801.81479

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

Academician A. N. KOLMOGOROV

ON THE LINEAR DIMENSIONALITY OF TOPOLOGICAL VECTOR SPACES

Two topological vector spaces E and E' are called **isomorphic** if a one-to-one linear correspondence, continuous in both directions, can be established between them.

It is well known that all topological vector spaces of one and the same finite dimension n are isomorphic to one another. In this trivial case the dimension of a space $\mathrm{d}(E)$ satisfies the requirements:

a) if E is isomorphic to a closed linear subspace of the space E', then $\mathrm{d}(E) \leq \mathrm{d}(E')$;

b) if E' is linearly and continuously mapped onto E, then $\mathrm{d}(E) \leq \mathrm{d}(E')$.

Banach, in Chapter XII of his well-known monograph ([1]), gave a generalization of the concept of dimension to the case of infinite-dimensional vector spaces, proceeding from the desire to preserve property a). In § 1 of the present note we introduce the linear dimension $\delta(E)$, satisfying both requirements a) and b). The resulting classification of spaces by dimension is, naturally, somewhat poorer than Banach' s, but in some respects more natural. For simplicity and for direct connection with Banach' s definition, all the spaces considered below will be assumed to be spaces of type F (see ([1]), Ch. III).

It may be considered well known how a metric of type F is introduced in the following spaces, which will subsequently be considered as examples:

$C_n^{(p)}$—the space of real functions $f(x_1, \dots, x_n)$, defined on the n-dimensional unit cube, having continuous partial derivatives up to order p inclusive, with the topology of uniform convergence together with the partial derivatives up to order p inclusive;

$B_n^{(\omega)}$—the space of real functions $f(x_1, \dots, x_n)$ of n real variables $x_1, \dots, x_n$, periodic with period 2π in each variable and having continuous partial derivatives of all orders, with the topology of uniform convergence both of the functions themselves and of their partial derivatives of arbitrary order;

A_n^G—the space of functions $f(z_1, \dots, z_n)$ of n complex variables $z_1, \dots, z_n$, analytic in a bounded open domain G of complex n-dimensional space, with the topology of uniform convergence on each compact set $K \subseteq G$.

SovietRxiv

The traditions and experience of classical analysis compel one to think that spaces of functions of a larger number of variables should be "richer" in elements than spaces of functions of a smaller number of variables: if the solution of a problem depends on an "arbitrary" function of one variable, then it is considered that there is less "arbitrariness" in the choice of the solution than if the solution depends on an arbitrary function of two variables, etc. We shall see below that in the case of **analytic** functions such notions find support in the corresponding inequalities of linear dimensions (Theorems 4 and 10). This result we also regard as the most interesting in

in the present note. On the contrary, for spaces of functions of finite smoothness the indicated expectation, based on the experience of classical analysis, is not confirmed in the properties of linear dimension! For example, from the results presented in Ch. XII of Banach' s monograph ([1]), it is easy to infer that all spaces $C_n^{(p)}$, independently of the values of n and p, have one and the same Banach dimension $\dim_l$. Since from the equality $\dim_l(E) = \dim_l(E')$ it always follows that $\delta(E) = \delta(E')$, the same applies also to the dimension δ introduced by us.

Passing to spaces of infinitely differentiable functions does not change the matter. For example, all spaces $B_n^{(\infty)}$ have one and the same dimension $\dim_l$ and one and the same dimension δ, since the following theorem is true:

Theorem 1. All spaces $B_n^{(\infty)}$, $n = 1, 2, ...$, are isomorphic. (See ([5]).)

§ 1. **Linear dimension $\delta(E)$.** Following Banach, we introduce the function δ, defining the meaning of the inequality

$$\delta(E) \leqslant \delta(E'). \tag{1}$$

By this the function δ is defined up to an order-preserving one-to-one mapping of the partially ordered set of its values Δ onto a new partially ordered set Δ'. By definition, (1) means that there exists a closed linear subspace E'' of the space E', linearly and continuously mappable onto the space E. The transitivity of the relation (1) thus defined, necessary for the correctness of the definition of the function, is easily proved.

We note here one property of the dimension $\delta(E)$ which is absent from the Banach dimension $\dim_l(E)$.

Theorem 2. If the spaces E and E' are Banach (of type B) and reflexive, then inequality (1) is equivalent to the inequality

$$\delta(\bar{E}) \leqslant \delta(\bar{E}') \tag{2}$$

between the dimensions of the conjugate spaces.

SovietRxiv

From Theorem 2 and the results of Ch. XII of Banach' s monograph ([1]), one can easily derive a number of results concerning the dimension $\delta(E)$ of Banach spaces; we shall not dwell on them. Instead, we formulate here several theorems on the dimension δ of spaces of analytic functions.

Theorem 3. If G and G' are two bounded finitely connected domains in the complex plane, then

$$\delta(A_1^G) = \delta(A_1^{G'}).$$

A generalization of Theorem 3 to functions of many variables has so far been proved only in the following form. Let $G_1, \dots, G_n$ be bounded finitely connected domains in the complex plane and $G = G_1 \times G_2 \times \dots \times G_n$; then the following theorem holds.

Theorem 3a. The dimension $\delta(A_n^G) = \alpha_n$ does not depend on the choice of the domains $G_1, G_2, \dots, G_n$.

For the dimension α_n introduced in Theorem 3a, Theorem 4 is valid.

Theorem 4. If $n < n'$, then $\alpha_n < \alpha_{n'}$.

The dimension $\delta(E)$ occupies an extreme position among all linear dimensions satisfying conditions a) and b).

Theorem 5. Any function $d(E)$ satisfying conditions a) and b) is representable in the form

$$d(E) = f[\delta(E)],$$

where from $\delta(E) \leqslant \delta(E')$ it follows that $d(E) \leqslant d(E')$.

Among such functions $d(E)$, subordinate to $\delta(E)$ and poorer in the sense of the possibility of distinguishing spaces by dimension, we shall consider

only one, which we shall call the "approximative dimension." With its help, Theorem 4 is proved, which is a direct consequence of Theorem 3a and of Theorem 10 given below.

§ 2. **Approximative dimension** $d_a(E)$. To each space E of type F we assign a class $\Phi(E)$ of functions $\varphi(\varepsilon)$, defined for $\varepsilon > 0$, by means of the condition $\varphi \in \Phi(E)$, if for every compact set $K \subset E$ and every open neighborhood U of the zero element θ in E there exists an ε_0 such that for every $\varepsilon < \varepsilon_0$ one can find $N \leq \varphi(\varepsilon)$ points $x_1, \dots, x_N$ of the space E such that

$$K \subset \bigcup_{1 \leq m \leq N} (x_m + \varepsilon U).$$

Two spaces E and E' have one and the same approximative dimension $d_a(E) = d_a(E')$, if $\Phi(E) = \Phi(E')$.

SovietRxiv

By definition, the inequality $d_a(E) < d_a(E')$ holds if $\Phi(E) \supset \Phi(E')$.

The relations $>, \leq, \geq, \|$ (incomparability) are defined analogously. The approximative dimension can in many cases be computed by methods adjoining the works devoted to ε-entropy and ε-capacity of metric spaces ([2–4]). We give some of the simplest results in this direction.

Theorem 6. *For an n-dimensional space E^n, with finite n, the set Φ is determined by the condition $\varphi \in \Phi$, if*

$$\lim_{\varepsilon \to \infty} (\varepsilon^n \varphi(\varepsilon)) = \infty.$$

Theorem 7. *For an infinite-dimensional Banach space E, the set Φ is empty.*

Thus all infinite-dimensional Banach spaces have the common dimension $d_a(E) = z$, which is maximal among the dimensions $d_a(E)$. It seems to us that this result should not be regarded as a circumstance compromising the dimension d_a. It is a meaningful concept for spaces that, in a certain sense, are closer to finite-dimensional ones; such are the countably normed spaces of the type B_n^∞ and A_n^G, which are acquiring ever greater importance in analysis.

Theorem 8. *The approximative dimension of the spaces does not depend on n and is determined by the condition: $\varphi \in \Phi$, if there exists a $q > 0$ such that*

$$\lim_{\varepsilon \to 0} (\varepsilon^q \log \varphi(\varepsilon)) = 0.$$

Theorem 9. *The approximative dimension d_s of the spaces A_s^G (G is an arbitrary bounded open domain of an s-dimensional complex space) does not depend on the choice of the domain G and is determined by the condition $\varphi \in \Phi$, if*

$$\lim_{\varepsilon \to 0} \frac{\log \varphi(\varepsilon)}{\left(\log \frac{1}{\varepsilon}\right)^{s+1}} = \infty.$$

Theorem 10 follows directly from Theorem 9.

Theorem 10. *If $s < s'$, then $a_s < a_{s'}$.*

Received
18 II 1958

References

[1] S. Banach, *Operations linéaires*, 1933.
[2] A. N. Kolmogorov, DAN, 108, No. 3 (1956).
[3] V. M. Tikhomirov, DAN, 117, No. 2 (1957).
[4] A. G. Vitushkin, DAN, 117, No. 5 (1957).
[4] A. Grothendieck, Mem. Am. Math. Soc., No. 16 (1955).

SovietRxiv

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.