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Abstract

Full Text

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ON A CERTAIN NECESSARY AND SUFFICIENT CRITERION FOR A FUNCTION TO BELONG TO THE SOBOLEV CLASS $W_p^{(1)}$

(Presented by Academician S. L. Sobolev, 19 IV 1958)

Definition. A continuous function $F(x)$ is said to have **bounded p-variation** on the interval $[a, b]$ if the sum

$$\sum_{i=0}^{n-1} \frac{|F(x_{i+1}) - F(x_i)|^p}{|x_{i+1} - x_i|^{p-1}}$$

is bounded above, for all partitions $a = x_0,\ x_1, \dots,\ x_n = b$, by a number V^p independent of n.

Riesz' s theorem ([1]). *In order that*

$$F(x) = \int_a^x f(\xi)\, d\xi,$$

where $f(x) \in L_p$ $(p > 1)$, it is necessary and sufficient that $F(x)$ have bounded p-variation.

This theorem establishes an isomorphism between the spaces $L_p(a, b)$ and $V_p(a, b)$—the functions with bounded p-variation that vanish at the point a—and in many questions of the theory of operations makes it possible to restrict oneself to the study of the class $V_p(a, b)$, which is simpler in certain respects (see, for example, ([2])).

We note that, by Riesz' s theorem, the class $V_p(a, b)$ coincides with the Sobolev class $W_p^{(1)}$ ([3]).

In what follows we shall carry out the discussion in two-dimensional space, in view of the complete analogy for the case of a larger number of dimensions.

Let Ω be a domain with boundary Γ such that the embedding theorems ([3]) are valid in it (for example, with smooth boundary). Consider a partition of Ω by the straight lines

$$x = x_i, \quad i = 0, \dots, m-1;$$

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$$y = y_j, \quad j = 0, \dots, n-1.$$

Let δ be the smallest distance between parallel straight lines of our partition.

Definition. We shall say that a continuous function $F(x, y)$, $(x, y) \in \Omega$, has **bounded p-variation** $(p > 2)$ if the sums

$$\sum_{i,j} \frac{|F(x_i + h, y_j) - F(x_i, y_j)|^p}{h^{p-2}}, \qquad \sum_{i,j} \frac{|F(x_i, y_j + h) - F(x_i, y_j)|^p}{h^{p-2}}$$

are bounded above by a number V^p independent of the partition and of h $(h \leq \delta/2)$. The sum is taken over the points (x_i, y_j) that lie in Ω together with their δ-neighborhood.

The definition of p-variation for a space of dimension greater than one appears, it seems, to be given for the first time. Its significance and its connection with Riesz' s theorem are shown by the following theorem.

Theorem. In order that a function belong to the class $W_p^{(1)}$, it is necessary and sufficient that it have bounded p-variation.

Proof. The sufficiency of the conditions follows at once from the weak convergence of the difference quotients to the corresponding partial derivatives and from the weak compactness of $W_p^{(1)}$.

To prove necessity, write the representation of the function in terms of its derivatives in the form (3)

$$u(P) = \frac{1}{|D|} \int_D u(Q)\, dQ - \int_D \sum_{i=1}^{n} \frac{B_i(P, Q)}{|P - Q|^{n-1}} \frac{\partial u(Q)}{\partial x_i}\, dQ,$$

which holds in any convex domain D of n-dimensional space, where $B_i(P, Q)$ are functions that do not increase when the domain is decreased, provided the ratio of the diameter to $|D|^{1/n}$ does not increase, and that are in this case bounded in absolute value independently of P, Q, and D.

We estimate the difference of the values of the function at neighboring points, for example $F(x_i+h, y_j) - F(x_i, y_j)$, using as the domain D a square K having a pair of opposite vertices at the points (x_i, y_j) and $(x_i + h, y_j)$ (from our assumptions on the partition it follows that these squares for different points will not intersect in a set of positive measure):

$$|F(x_i + h, y_j) - F(x_i, y_j)| \leq 2M \int_K \frac{|\operatorname{grad} F(\xi, \eta)|\, d\xi\, d\eta}{\sqrt{(x_i - \xi)^2 + (y_j - \eta)^2}},$$

where $M = \sup_{P,Q} B_i(P, Q)$, and M does not depend on the partition.

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Applying Hölder' s inequality with exponents p and p' and passing to polar coordinates, we finally obtain the estimate

$$|F(x_i+h,y_j)-F(x_i,y_j)| \leq C\left\{\iint_K |\operatorname{grad} F|^p\,dx\,dy\right\}^{1/p} h^{2/p'-1},$$

$$|F(x_i,y_j+h)-F(x_i,y_j)| \leq C\left\{\iint_{K'} |\operatorname{grad} F|^p\,dx\,dy\right\}^{1/p} h^{2/p'-1},$$

where C does not depend on the partition, the point x_i, y_j, or the function $F(x,y)$.

Raising to the p-th power and summing over i, j, we obtain the required inequality

$$\sum_{i,j}\frac{|F(x_i+h,y_j)-F(x_i,y_j)|^p}{h^{p-2}}+\sum_{i,j}\frac{|F(x_i,y_j+h)-F(x_i,y_j)|^p}{h^{p-2}} \leq C_1\|F\|_{W_p^{(1)}}.$$

Remark 1. With more careful estimates one can obtain $C_1 = 1$.

Remark 2. For arbitrary n the proof remains valid if one takes $p > n$ and replaces the square K by a pair of pyramids with a common base.

Remark 3. The present theorem strengthens Kondrashov' s result ([3]) on the uniform Hölder condition of order $1-n/p$ for functions of the class $W_p^{(1)}$.

The fact that Hölder conditions of this order are not sufficient for membership in the class $W_p^{(1)}$ is shown by the example of the function $F(x,y)=\sqrt[4]{(x^2+y^2)}$ in the disk $x^2+y^2<1$, which has Hölder exponent equal to 1/2, but does not belong to the class $W_4^{(1)}$. In the present note it is proved that invertibility holds if the Hölder inequalities can be summed over points lying at the vertices of non-overlapping squares.

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Note: Figure translations are in progress. See original paper for figures.

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