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Abstract

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ZONAL SPHERICAL FUNCTIONS AND LAPLACE OPERATORS ON SOME SYMMETRIC SPACES

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Let $\mathfrak{M} = G/H$ be a homogeneous space, whose stationary subgroup H we shall assume to be compact. Consider on $\mathfrak{M}$ a differential operator Δ that commutes with the translation operators $f(M) \to f(gM)$. Such differential operators form a ring and are called **Laplace operators on** $\mathfrak{M}$ ([1]). Let R be the set of functions on $\mathfrak{M}$ that are constant on the transitivity surfaces of the subgroup H. (It is clear that R may be interpreted as the set of functions constant on the double cosets of the group G with respect to H.) If Δ is a Laplace operator and $f \in R$, then $\Delta f \in R$. Consequently, every Laplace operator induces a certain differential operator on R, which we call the **radial part of the operator** Δ and denote by $\overset{0}{\Delta}$. In the present paper the radial parts of Laplace operators and the zonal spherical functions belonging to irreducible representations are computed for certain symmetric spaces, namely: for the space $\mathfrak{M}^+_{n,k}$ $(n \geq 2k)$– the manifold of k-dimensional subspaces of an n-dimensional complex space (the so-called complex Grassmann spaces), and also for two other symmetric spaces $\mathfrak{M}^-_{n,k}$ and $\mathfrak{M}^0_{n,k}$, which are closely connected with $\mathfrak{M}^+_{n,k}$ and will be described below.

If the space $\mathfrak{M}$ is symmetric, then the spherical functions φ belonging to irreducible representations of the group G satisfy the system of equations ([1])

$$\Delta \varphi^n_i = \lambda(\Delta)\varphi, \tag{1}$$

where $\Delta \to \lambda(\Delta)$ is a homomorphism of the ring of Laplace operators into the field of complex numbers.

In the case when φ is a zonal spherical function, the system (1) can be rewritten in the form

$$\overset{0}{\Delta}\varphi = \lambda(\Delta)\varphi. \tag{2}$$

Let us now note that if the group G of motions of the symmetric space $\mathfrak{M}$ is semisimple, then every element g of G is representable in the form

$$g = h_1 \varepsilon h_2, \tag{3}$$

where h_1 and h_2 are in H, and ε belongs to some commutative subgroup E ([2]). The set of inner automorphisms $g \to hgh^{-1}$ $(h \in H)$ of the group G that carry E into itself induces on E some finite group S of automorphisms of E. Formula (3) assigns to each element g of G a finite number of elements ε of E. The elements ε_1 and ε_2 correspond, by formula (3), to one and the same element g if and only if their squares

can be transformed into one another by means of the group S. Hence it is clear that every function from R, i.e., a function constant on double cosets with respect to H, may be regarded as a function of ε^2, and moreover one such that $f(\varepsilon^2) = f(s\varepsilon^2)$ for every s in S. It also follows from what has been said that the operator $\overset{0}{\Delta}$ may be interpreted as a differential operator on E.

Let now $\mathfrak{M}^+_{n,k}$ be the complex Grassmann space. The group of motions of $\mathfrak{M}^+_{n,k}$ is the group $SU(n)$ of unimodular unitary transformations of the n-dimensional complex space; the stationary subgroup H is the subgroup of $SU(n)$ leaving invariant a certain k-dimensional subspace. In the group E one can introduce canonical coordinates $t_1, \dots, t_k$ so that the group S will consist of all possible permutations of the variables $t_1, \dots, t_k$ and all possible changes of sign of them.*

In the coordinates $t_1, \dots, t_k$, the radial parts of the Laplace operators on $\mathfrak{M}^+_{n,k}$ are written in the form

$$\overset{0}{\Delta} = \frac{1}{j(t)} P(L_1, \dots, L_k)\, j(t), \tag{4}$$

where

$$j(t) = \prod_{p<q} \left(\sin^2 t_p - \sin^2 t_q\right),$$

$$L_p = \frac{1}{4}\frac{\partial^2}{\partial t_p^2} + \frac{1}{2}\left[\operatorname{ctg} 2t_p + (n-2k)\operatorname{ctg} t_p\right]\frac{\partial}{\partial t_p},$$

$P(x_1, \dots, x_k)$ is an arbitrary symmetric polynomial in $x_1, \dots, x_k$.

Solving system (2) and taking into account that the zonal spherical function φ must be a polynomial in e^{2it_p} and possess the symmetry properties indicated earlier, we obtain**

$$\varphi = \frac{d(r_1, \dots, r_k)}{j(t)}\, \lambda(r_1, \dots, r_k), \tag{5}$$

where

$$d(r_1, \dots, r_k) = \det \left\| J^m_{r_j}(x_i) \right\|; \qquad m = n - 2k; \qquad x_i = \sin^2 t_i;$$

$$J^m_r(x) = F(m + r + 1, -r, m + 1; x);$$

$F(\alpha, \beta, \gamma; x)$ is the hypergeometric function;

$$\lambda(r_1, \dots, r_k) = \frac{1}{\prod\limits_{i<j} (\rho_i - \rho_j)} \prod_{s=1}^{k-1} s!\, (m + s)^{\,k-s}; \qquad \rho_i = -r_i(m + 1 + r_i).$$

Let us note that the functions $J^m_r(x)$ coincide, up to a constant factor, with the Jacobi polynomials $P_r^{(m,0)}(\cos 2t)$. The numbers $r_1, \dots, r_k$ are connected with the homomorphism $\Delta \to \lambda(\Delta)$ and with the highest weight of the representation to which φ belongs, as follows: if Δ is given by formula (4), then

$$\lambda(\Delta) = P(\rho_1, \dots, \rho_k), \qquad r_i = p_i + k - i, \tag{6}$$

where $p_1, \dots, p_k$ are the numbers defining the representation. (If $l_1 \geq l_2 \geq \dots \geq l_n$ are the coordinates of the highest weight of the representation according to Cartan, and $\sum l_i = 0$, then $p_i = l_i = -l_{n-i+1}$, $i = 1, \dots, k$, while the remaining l_i are equal to 0.) From formula (6)

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* $e^{\pm it_1}, \dots, e^{\pm it_k}$ are the characteristic roots of the transformation from E (the remaining roots are equal to 1).

** The function φ is normalized so that $\varphi(0) = 1$.

it is seen that the homomorphism $\Delta \to \lambda(\Delta)$ determines the highest weight of the representation realized in spherical functions. Formula (5) for the function φ may also be obtained by orthogonalizing the sequence of symmetric polynomials in $\sin^2 t_1, \dots, \sin^2 t_k$ with weight equal to

$$j(t)^2 \prod |\sin 2t_i|\, \sin^2 t_i.$$

(If the group G is compact, then the zonal spherical functions belonging to irreducible representations form a complete orthogonal system of functions in

R with the weight induced by the invariant measure in $\mathfrak{M}$. For the space $\mathfrak{M}^+_{n,k}$ this weight is equal to

$$j(t)^2 \prod |\sin 2t_i| \sin^2 t_i.$$

)

Other spaces for which the radial parts of the Laplace operators and the zonal spherical functions are computed are the space $\mathfrak{M}^-_{n,k}$, Cartan-dual to $\mathfrak{M}^+_{n,k}$, and the space $\mathfrak{M}^0_{n,k}$, which may be regarded as a limiting case of $\mathfrak{M}^+_{n,k}$ and $\mathfrak{M}^-_{n,k}$.

The group G of motions of the space $\mathfrak{M}^-_{n,k}$ is the group of all unimodular complex matrices of order n that preserve the form

$$\sum_1^k z_i \bar{z}_i - \sum_{k+1}^n z_i \bar{z}_i.$$

The stationary subgroup $H = G \cap SU(n)$ evidently coincides with the stationary subgroup of the space $\mathfrak{M}^+_{n,k}$. The formulas for the Laplace operators on $\mathfrak{M}^-_{n,k}$ and for the zonal spherical functions may be obtained from formulas (4) and (5), respectively, if in them t is replaced by $\sqrt{-1}\, t$.

The relation between the homomorphism $\Delta \to \lambda(\Delta)$ and the numbers $r_1, \dots, r_k$ is still given by formula (6). From formula (6) it is seen that the homomorphism $\Delta \to \lambda(\Delta)$ uniquely determines the system of numbers $(\rho_1, \dots, \rho_k)$, considered up to all possible permutations. In turn, the system $(\rho_1, \dots, \rho_k)$ uniquely determines, by formula (5), the function φ, and hence also the representation realized in spherical functions. For arbitrary complex $\rho_1, \dots, \rho_k$ the corresponding representation is a representation by means of bounded operators in a Banach space. For real integral $r_1, \dots, r_k$ this representation is finite-dimensional. If the system of numbers $(\rho_1, \dots, \rho_k)$ coincides with the system of numbers $(\bar{\rho}_{i_1}, \dots, \bar{\rho}_{i_k})$ $(i_1, \dots, i_k$ is some permutation of the indices $1, \dots, k)$, then the representation admits an invariant Hermitian form, perhaps not positive definite.

The space $\mathfrak{M}^0_{n,k}$ is the space of all complex matrices with k rows and $n-k$ columns. The motions in $\mathfrak{M}^0_{n,k}$ are given by the formula $A \to U_k A U^{-1}_{n-k} + B$, where U_k and U_{n-k} are arbitrary unitary matrices of orders k and $n-k$, respectively, connected by the relation

$$\det U_k \cdot \det U_{n-k} = 1,$$

and A and B are in $\mathfrak{M}^0_{n,k}$. Although the group of motions of the space $\mathfrak{M}^0_{n,k}$ is not semisimple, for $\mathfrak{M}^0_{n,k}$ a number of assertions of the theory of symmetric spaces with a semisimple group of motions hold. In particular, the spherical functions on $\mathfrak{M}^0_{n,k}$ are eigenfunctions for the Laplace operators. Every element of $\mathfrak{M}^0_{n,k}$ can be represented in the form $U_k E U^{-1}_{n-k}$, where E is a matrix in $\mathfrak{M}^0_{n,k}$ of the form

$$E = \begin{pmatrix} t_1 & & 0 \\ & \cdot & \\ 0 & & \\ & t_k\, 0 \dots 0 & \end{pmatrix},$$

where the numbers $t_1, \dots, t_k$ are real and are determined up to an arbitrary permutation and an arbitrary change of sign. Therefore,

the operators $\overset{0}{\Delta}$ are operators in the space of symmetric even functions of $t_1, \dots, t_k$.

The radial part of the Laplace operators on $\mathfrak{M}^0_{n,k}$ has the form

$$\overset{0}{\Delta} = \frac{1}{j(t)}\, P(L_1, \dots, L_k)\, j(t), \tag{7}$$

where

$$j(t) = \prod_{p<q}(t_p^2 - t_q^2); \qquad L_p = \frac{\partial^2}{\partial t_p^2} + \frac{1+2m}{t_p}\frac{\partial}{\partial t_p}; \quad m = n - 2k;$$

$P(x_1, \dots, x_k)$ is an arbitrary symmetric polynomial in $x_1, \dots, x_k$.

The zonal spherical function φ has the form*

$$\varphi = \frac{d(r_1, \dots, r_k)}{j(t)}\, \lambda(r_1, \dots, r_k), \tag{8}$$

where

$$d(r_1, \dots, r_k) = \det \|P(r_i t_j)\|; \quad P(x) = \frac{1}{x^m} J_m(x);$$

$J_m(x)$ is the Bessel function;

$$\lambda = \frac{m!(m+1)! \dots (m+k-1)!\, 2^{k(k-1)}}{\prod_{i<j}(\rho_i - \rho_j)}; \qquad \rho_i = -r_i^2.$$

The numbers $r_1, \dots, r_k$ are connected with the homomorphism $\Delta \to \lambda(\Delta)$ in the following way. If the operator is given by formula (7), then

$$\lambda(\Delta) = P(\rho_1, \dots, \rho_k). \tag{9}$$

From formulas (8) and (9) it is clear that the representation is specified by the system of numbers $(\rho_1, \dots, \rho_k)$, considered up to all possible permutations. For real $\rho_1, \dots, \rho_k$ this representation admits an invariant Hermitian form.

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* The function φ is normalized so that $\varphi(0) = 1$.

Note: Figure translations are in progress. See original paper for figures.