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Abstract

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MATHEMATICS

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PROOF OF A THEOREM ON THE HOMEOMORPHISM OF POLYHEDRA AND POINT SETS

(Presented by Academician P. S. Aleksandrov on V 6, 1958)

1. Preliminary concepts and statement of the question.

P. S. Aleksandrov introduced the following definitions ([1,2]).

A **triangulation** lying in a given Euclidean space R^n is any set τ of pairwise nonintersecting open simplexes (of different numbers of dimensions) of the space R^n, satisfying the following conditions: 1) every face of a simplex that is an element of the set τ is itself an element of this set (the condition of "completeness"); 2) every point x belonging to some simplex $t \in \tau$ has a neighborhood (relative to R^n) that intersects only finitely many simplexes from τ. From these conditions it follows that every triangulation consists of a finite or countable number of elements; condition 2), obviously, is substantive only for infinite triangulations.

The set-theoretic sum of all simplexes belonging to a given triangulation τ is called the **body** of this triangulation and is denoted by $\tilde{\tau}$. A set $A \subseteq R^n$ that is the body of some triangulation is called a **polyhedron**. It is known and easily proved that every polyhedron is a locally compact space; only the bodies of finite triangulations ("finite polyhedra") are compact.

Fundamental for what follows is the concept of following, which makes it possible to regard any set of triangulations lying in a given R^n as partially ordered: we say that a **triangulation** τ' **follows the triangulation** τ, and write $\tau' \geq \tau$, if every simplex $t' \in \tau'$ is contained in some (obviously unique) simplex $t \in \tau$ (its "carrier" in τ).

If $\tau' \geq \tau$, then, assigning to each simplex $t' \in \tau'$ its carrier $t \in \tau$, which we shall denote by $\mathfrak{F}_{\tau}^{\tau'} t'$, we obtain a mapping $\mathfrak{F}_{\tau}^{\tau'}$ of the triangulation τ' into the triangulation τ. This mapping is called the **geometric projection** of the triangulation τ' onto the triangulation τ.

Let now $A \subseteq R^n$ be some polyhedron (finite or infinite), given together with

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some triangulation τ_0 of it, so that $A = \tilde{\tau}_0$. Then the set of all triangulations that are subdivisions of the triangulation τ_0 is also given; this set, by virtue of the definition of following established by us, is partially ordered and even directed, and in it projections $\mathfrak{F}_{\tau}^{\tau'}$ are defined for any $\tau' \geq \tau$.

Definition 1. The directed set of all subdivisions of the given triangulation τ_0 of the polyhedron $A = \tilde{\tau}_0$, together with the projections $\mathfrak{F}_{\tau}^{\tau'}$ defined in it, is called the **combinatorial spectrum** of the triangulated polyhedron $A = \tilde{\tau}_0$.

Definition 2. Let A be an arbitrary set lying in R^n. A triangulation τ **covers** the set A if $A \subseteq \tau$. The directed set of all triangulations covering the set A, together with the projections $\mathfrak{D}_{\tau}^{\tau'}$ defined above, is called the **geometric spectrum** of the set A.

Finally, let us introduce the definition of an abstract spectrum. For this, we define generalized simplicial mappings of one abstract complete star-finite simplicial complex β into another, α. Let to each vertex e_β of the complex β there correspond a certain simplex

$$t_\alpha = \mathfrak{D}_\alpha^\beta e_\beta$$

of the complex α in such a way that, if the vertices $e_{\beta 1}, \dots, e_{\beta r}$ form the star of some simplex $t_\beta \in \beta$, then the simplexes

$$\mathfrak{D}_\alpha^\beta e_{\beta 1}, \dots, \mathfrak{D}_\alpha^\beta e_{\beta r}$$

have in α a combinatorial sum, i.e. are faces of some simplex of the complex α. This combinatorial sum itself, i.e. the simplex $t_\alpha \in \alpha$ whose star is the set-theoretic sum of the stars of the simplexes

$$\mathfrak{D}_\alpha^\beta e_{\beta 1}, \dots, \mathfrak{D}_\alpha^\beta e_{\beta r},$$

is denoted by $\mathfrak{D}_\alpha^\beta t_\beta$ and is defined as the image of the simplex

$$t_\beta = |e_{\beta 1} \dots e_{\beta r}|$$

under the mapping $\mathfrak{D}_\alpha^\beta$. Mappings defined in this way are called **generalized simplicial mappings**.

Definition 3. An **abstract spectrum** is a directed set Σ of abstract star-finite simplicial complexes $\alpha, \beta, \dots$ together with projections, i.e. generalized simplicial mappings $\mathfrak{D}_\alpha^\beta$, defined for any pair of complexes α, β for which $\beta > \alpha$, where $\mathfrak{D}_\alpha^\beta$ is a mapping of the complex β into the complex α, and for any $\gamma > \beta > \alpha$ from Σ the following condition of "**weak transitivity**" is assumed to hold: for an arbitrary vertex $e_\gamma \in \gamma$ the simplex

$$\mathfrak{D}_\alpha^\beta \mathfrak{D}_\beta^\gamma e_\gamma$$

is a proper or improper face of the simplex

$$\mathfrak{D}_\alpha^\gamma e_\gamma.$$

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Remark 1. Identifying the simplexes of an abstract complex with their stars (which we shall constantly do below), we may write the condition of weak transitivity simply in the form

$$\mathfrak{D}_\alpha^\beta \mathfrak{D}_\beta^\gamma e_\gamma \subseteq \mathfrak{D}_\alpha^\gamma e_\gamma.$$

Remark 2. For the combinatorial and geometric spectra even the condition of strong transitivity is fulfilled:

$$\mathfrak{D}_\alpha^\beta \mathfrak{D}_\beta^\gamma e_\gamma = \mathfrak{D}_\alpha^\gamma e_\gamma.$$

Spectra satisfying this condition are called **transitive**.

Having introduced these definitions, P. S. Aleksandrov, in the same works ([1],[2]), poses the following problems:

Under what necessary and sufficient conditions imposed on the combinatorial spectra of two polyhedra, respectively on the geometric spectra of two sets $A \subseteq R^n$, $B \subseteq R^m$, can one assert that these polyhedra (respectively these sets) are homeomorphic to each other?

The answer requires the introduction of one more concept—the concept of cofinality. We say that a spectrum σ is a **cofinal part of a spectrum** Σ, or that a spectrum σ is **cofinally complemented to a spectrum** Σ, if the directed set of complexes constituting the spectrum σ is a cofinal part (in the ordinary sense of the word) of the directed set of complexes constituting the spectrum Σ, and if the projections $\mathfrak{D}_\alpha^\beta$ in σ are the same as in Σ.

After this we can, together with P. S. Aleksandrov, formulate the following main theorem.

Theorem on homeomorphism. *In order that two sets $A \subseteq R^n$ and $A' \subseteq R^{n'}$ be homeomorphic, it is necessary and sufficient that*

their geometric spectra S in R^n and S' in $R^{n'}$ should have confinal parts s and s', which can be confinally completed to one and the same abstract spectrum Σ.

However, P. S. Aleksandrov proves in ([1,2]) a theorem which, although it is worded exactly the same as the one just formulated, uses the concept of confinality not in the simple and natural sense in which we understand it in the present note, but in another, considerably more complicated sense. Therefore P. S. Aleksandrov, in § 7 of his work ([2]), poses the problem of proving the homeomorphism theorem precisely with this simple understanding of the term confinality. This problem is solved in the present note. We give a brief exposition of the proof.

2. Proof of the homeomorphism theorem; necessity. By an open spectrum of a given space A we mean the set of nerves of all star-finite open coverings of the space A, with the natural order: $\beta > \alpha$ if the covering β is inscribed in

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the covering α. (P. S. Aleksandrov requires the so-called regular inscription of the covering β in α, which, however, is superfluous.) The projections are defined as usual: for $e_\beta \in \beta$ the simplex $\delta_\alpha^\beta e_\beta$ is defined as the simplex determined by all e_α containing the set e_β. Canonical coverings and the canonical spectrum are defined in the same way as in P. S. Aleksandrov, and it is likewise proved that some multiplication of the open spectrum of a set $A \subseteq R^n$ contains, as a confinal part, the canonical spectrum of this set, which in turn is a confinal part of its geometric spectrum. The necessity of the condition contained in the homeomorphism theorem follows at once from this.

3. Proof of the homeomorphism theorem; sufficiency. The plan of the proof is the same as in P. S. Aleksandrov: the same notion of the space of a given spectrum is introduced; it is proved that, in passing from a given spectrum to its confinal part, the space does not change; and, finally, it is proved that the geometric spectrum of a given set A has this set as its space. The sufficiency clearly follows from this. However, since P. S. Aleksandrov uses a different notion of confinality from ours, we must prove anew:

Basic proposition. *The space of any spectrum is homeomorphic to the space of each of its confinal parts.*

We note that in our proof we make no use whatsoever of the concept of a "projection set" introduced by P. S. Aleksandrov (see ([2]), § 1), but reason only about threads (see ([2]), § 3) of the given spectrum. It is apparently precisely the use of the concept of a projection set as an intermediate stage in the argument that caused the need for the complicated notion of confinality, from which we are now freeing ourselves.

The basic proposition is contained in the combination of the following two lemmas:

Lemma 1. *Leaving in a thread of the spectrum Σ only the vertices belonging to complexes $\alpha' \in \sigma$, we obtain a thread of the spectrum σ.*

Lemma 2. *Completing a thread ξ' of the spectrum σ by all projections of its elements, we obtain a set ξ which is a thread of the spectrum Σ.*

Lemma 1 is proved without difficulty by direct verification. For the proof of Lemma 2, consider an arbitrary finite subset $e_{\alpha_1}, \dots, e_{\alpha_s}$ of the set ξ. If $e_{\alpha_i} \in \xi'$, put $\alpha_i' = \alpha_i,\ e_{\alpha_i'} = e_{\alpha_i}$.

If $e_{\alpha_j} \notin \xi'$, take some $e_{\alpha_i'} \in \alpha_i' > \alpha_i$ subject to the condition $e_{\alpha_j} \in \delta_{\alpha_i}^{\alpha_i'} e_{\alpha_i'}$.

As a result we obtain elements $e_{\alpha_1'}, \dots, e_{\alpha_s'}$ of the thread ξ'. For them we choose an $\alpha_0' \in \sigma$ such that for any $e_{\alpha_0'} \in \xi' \cap \alpha_0'$ it is true that

$$e_{\alpha_1'} \in \mathfrak{D}_{\alpha a' 1}^{\alpha_0'} e_{\alpha_0'}, \dots, e_{\alpha_s'} \in \mathfrak{D}_{\alpha a' s}^{\alpha_0'} e_{\alpha_0'}.$$

Then

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$$e_{\alpha_i} \in \mathfrak{D}_{\alpha_i}^{\alpha'_i} e_{\alpha'_i} \subseteq \mathfrak{D}_{\alpha_i}^{\alpha'_i} \mathfrak{D}_{\alpha a' i}^{\alpha'_0} e_{\alpha'_0} \subseteq \mathfrak{D}_{\alpha_i}^{\alpha'_0} e_{\alpha'_0}, \qquad i = 1, 2, \ldots, s,$$

whence it follows that ξ is a thread of the spectrum Σ.

The theorem on homeomorphism is thereby proved.

The condition for homeomorphism for two triangulated polyhedra is obtained from the general theorem on homeomorphism by replacing, in its formulation, geometric spectra by combinatorial ones.

The following problem (communicated to me by P. S. Aleksandrov) is, apparently, very difficult.

Does the theorem on homeomorphisms remain valid (even in the particular case of polyhedra) if, in its formulation, by a spectrum one always understands only a transitive spectrum?

We arrive at this problem because both geometric and combinatorial spectra are transitive; therefore it is natural to require transitivity also of a spectrum that is a cofinal refinement of the cofinal parts of geometric or combinatorial spectra.

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CITED LITERATURE

[1] P. S. Aleksandrov, DAN, **97**, No. 5, 757 (1954).
[2] P. S. Aleksandrov, Tr. Mosk. matem. obshch., **4**, 405 (1955).

Note: Figure translations are in progress. See original paper for figures.

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