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Abstract

Full Text

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AXISYMMETRIC PLANE ELASTIC-FRIABLE PROBLEM OF THE PROPAGATION OF A STATE OF LIMIT EQUILIBRIUM

(Presented by Academician S. L. Sobolev on 5 IX 1957)

§ 1. Formulation of the problem. Hypotheses of motion.

We consider the plane strained state for a medium filling, in the xy-plane, the exterior of a unit circle. It is known that under small stresses the medium is described by the equations of the linear theory of elasticity, while under large stresses it acquires friable properties ([2]).

Beginning from the moment $t = 0$, the cut-out begins to expand according to the law $r = r_1(t)$, $r_1(0) = 1$, and to rotate according to the law $\psi = \psi_1(t)$, $\psi_1(0) = 0$ (ψ_1 is the angle of rotation of the circumference of the cut-out). It is required to determine the stress and displacement in the elastic and friable zones, the boundary $\gamma(t)$ between the zones, and to investigate the admissibility of the boundary conditions $\{r_1(t), \psi_1(t)\}$ for the preservation of equilibrium on the boundary. The formulation of the problem is a complete analogue of formulation ([1]) for a plastic medium.

In the solution two hypotheses of motion were used: the Mises hypothesis ([3,4]) and the author's hypothesis, which consists in the fact that, while preserving the incompressibility condition, the coincidence is assumed of one direction of the maximum shear velocity with one direction of the slip lines, namely with that which on the contour has the smaller angle with the boundary. The problem without rotation in our hypothesis proves to be unstable: in the presence of a small rotation the motion differs strongly from pure expansion.

§ 2. Pure expansion. Dilatancy.

The problem with pure expansion is solved explicitly under both hypotheses. However, solution of the problem under the Mises hypothesis with the aid of equations ([4]) leads to the paradoxical conclusion of unlimited dilatancy of the medium for large values of the radius of the cut-out. The increment ΔV of a small volume V on the contour is given by the formula

$$\Delta V = V \sin\rho \cdot \frac{1+\alpha}{2} \ln r_1(t),$$

where α is a positive function, ρ is the angle of friction ([2]).

§ 3. Problem with rotation

The problem with rotation reduces to the integration in the friable zone of the system of equilibrium equations

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\vartheta}{r} = 0, \qquad \frac{d\tau_{r\vartheta}}{dr} - \frac{2\tau_{r\vartheta}}{r} = 0 \tag{1}$$

with allowance for the equation of state

$$(\sigma_r - \sigma_\vartheta)^2 + 4\tau_{r\vartheta}^2 = (\sigma_r + \sigma_\vartheta + 2k \operatorname{ctg} \rho)^2. \tag{2}$$

On the contour the displacements are prescribed: $u_r = r_1(t) - r_*$, $u_\vartheta = \psi_1(t) r_1(t)$; in the elastic zone the solution is given by formulas (35), (36) of work ([1]).

The extension to the case of a friable medium of Sobolev' s hypothesis on the minimality of the plastic zone ([1]) leads to continuity of the stress tensor on the boundary of the zones.

The displacement velocity v_ϑ, obtained under our hypothesis of motion, is found from the solution of the differential equation

$$\frac{dv_\vartheta}{dr} - \frac{v_\vartheta}{r} + \frac{E'(t)}{r^2} \tan(2\varphi - \rho) = 0, \tag{3}$$

where $E(t)$ is a known function of time; φ is the angle between the first principal axis of the stress tensor and the radius, and depends on r and t.

Integrating (3), we obtain v_ϑ, containing an arbitrary constant (with respect to r) depending on time.

Using the continuity of the stress, displacement, and velocity tensors on $\gamma(t)$, we obtain, for determining the unknown functions of time, a closed system of equations.

All the unknown functions are expressed in finite form through $A(t)$, which is the solution of the Cauchy problem for the integro-differential equation

$$\psi'(t) = \frac{k}{2\mu} \left(\frac{\cos\rho \cdot A'(t)}{\sqrt{A^2 + C^2}} + 2C' \int_{r_1(t)}^{-\frac{1}{\sqrt{\cos\rho}} \sqrt[4]{A^2+C^2}} \frac{\tan(2\varphi - \rho)}{\xi^3} \, d\xi \right). \tag{4}$$

Here φ is a known function of $A(t)$, t, and r, and $C(t)$ is a known function of time. Existence and uniqueness of the solution are proved by the method of contraction mappings.

Using the Mises hypothesis and equations of work[4], for $A(t)$ we obtain the equation

$$\psi'(t) = \frac{k}{2\mu}\cos\rho\left[\frac{A'(t)}{\sqrt{A^2+C^2}} - \frac{C'}{A}\left(\cos 2\varphi_\gamma - \cos 2\varphi_1\right)\right], \qquad (4')$$

for which the solution of the Cauchy problem likewise exists and is unique.

§ 4. **Investigation of the admissibility of the motion of the contour.** Investigating equations (4) and (4′) analogously to how this is done in [1], one can prove that $\psi_1(t)$ cannot grow at a rate independent of the growth of $r_1(t)$. The maximum possible growth of $\psi_1(t)$ for large $r_1(t)$ is given, under both hypotheses, by a formula of the form

$$\psi_1(t) \sim M(\rho)\ln r_1(t),$$

where $M(\rho)$ is a known function, satisfying under both hypotheses the conditions

$$M(0) = 1, \qquad M\left(\frac{\pi}{2}\right) = 0, \qquad \frac{dM(\rho)}{d\rho} < 0.$$

The physical meaning of the maximum $\psi_1(t)$ is that, when the contour rotates with this velocity, the slip lines are tangent to the contour. When the velocity is increased, rupture of the medium occurs on the contour along the slip lines[1].

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CITED LITERATURE

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[2] V. V. Sokolovskii, *Statics of Granular Media*, 1954.
[3] H. Geiringer, in the collection *Problems of Mechanics*, Foreign Literature Publishing House, 1955, p. 150.
[4] V. V. Sokolovskii, *Applied Mathematics and Mechanics*, **19**, issue 1 (1955).

Note: Figure translations are in progress. See original paper for figures.