CRITICAL SECTIONS IN IRREGULAR WAVEGUIDES

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Abstract

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B. Z. KATSENELENBAUM

CRITICAL SECTIONS IN IRREGULAR WAVEGUIDES

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1. In [1] the amplitude of the wave H_{0n}, arising when a wave H_{01} is incident on a symmetric waveguide transition in some section of which the wave number h_n of the wave H_{0n} becomes zero, was calculated. Below the same problem is solved for any wave in a rectilinear irregular hollow waveguide of arbitrary shape with ideal walls.

The electromagnetic field in an irregular waveguide is described, for time dependence $\exp(i\omega t)$, by the infinite system of equations

$$P_i' + ih^i P_i = \sum_{\nu=-\infty}^{\infty} S_{i\nu} P_\nu. \tag{1}$$

(the prime denotes differentiation with respect to the coordinate z along the waveguide) and by boundary conditions at the ends of the irregular segment, $z = 0$ and $z = L$; these conditions are determined by which wave is incident on the irregular segment. In (1), $P_i(z)$ are the amplitudes of waves of different types that can exist in a regular waveguide with the same cross-section as the section of the irregular waveguide at the given z; $i > 0$ corresponds to forward waves, i.e., waves propagating in the $+z$ direction, and $i < 0$ to backward waves. The general expression for the coupling coefficients $S_{im}(z)$ is given in [2]. For the various possible combinations of electric and magnetic waves it gives (for $i \neq m$):

$$S_{im} = \frac{1}{2h^i(h^i - h^m)} \oint \vartheta \left\{ (\alpha^i)^2 (\alpha^m)^2 \psi^i \psi^m + (h^i h^m - k^2) \frac{\partial \psi^i}{\partial s} \frac{\partial \psi^m}{\partial s} \right\} ds,$$

$$S_{im} = -\frac{k}{2h^i} \oint \vartheta \frac{\partial \varphi^i}{\partial n} \frac{\partial \psi^m}{\partial s}\, ds, \qquad S_{im} = \frac{k^2 - h^i h^m}{2h^i(h^i - h^m)} \oint \vartheta \frac{\partial \varphi^i}{\partial n} \frac{\partial \varphi^m}{\partial n}\, ds. \tag{2}$$

Here the integrals are taken over the contour of the cross-section; $\vartheta(s, z)$ is the tangent of the angle formed by the tangent to the waveguide, perpendicular to the contour of the cross-section, and the z-axis. The functions ψ^i, φ^i describe the fields of magnetic and electric waves; they satisfy identical equations

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \alpha^2)\psi = 0$$

and are normalized by the condition

$$\int \psi^2\, dS = \alpha^{-2},$$

where the integral is taken over the cross-section. On the contour $\varphi^i = 0$, $\partial\psi^i/\partial n = 0$; by definition $\varphi^{-i} = \varphi^i$, $\psi^{-i} = -\psi^i$. The quantities

$$h^i(z) = \{k^2 - (\alpha^i)^2\}^{1/2}$$

are the wave numbers in the waveguide, $h^{-i} = -h^i$; k is the wave number in vacuum.

The coupling coefficient S_{ii} is equal to $-(h^i)'/2h^i$ ([2]). Applying the same methods as in ([2]), we obtain for magnetic waves

$$S_{ii} = -\frac{(h^i)'}{2h^i} = -\frac{(\alpha^i)^2}{4(h^i)^2} \oint \vartheta \left\{ (\alpha^i)^2(\psi^i)^2 - \left(\frac{\partial \psi^i}{\partial s}\right)^2 \right\} ds. \qquad (3)$$

For electric waves in the last formula the expression in the braces is to be replaced by $(\partial\varphi^i/\partial n)^2$. Formulas (2), (3) correspond to formulas (8), (35), (10), (11)

and (37) of paper ([3]). For smooth transitions, for which $\vartheta_0 \ll 1$, where ϑ_0 is a certain mean value of $|\vartheta(s, z)|$, the coefficients S_{im} are small. In the leading order in ϑ_0, in system (1) it is necessary to retain, for each i, only two terms; in this case the system takes the form

$$P_i' + ih^i P_i = S_{ii}P_i + S_{i1}P_1, \qquad P_{-i}' + ih^{-i}P_{-i} = S_{-i,-i}P_{-i} + S_{-i,1}P_1. \qquad (4)$$

Here P_1 is the amplitude of the incident wave; we shall take $P_1(0) = 1$.* If the wave is incident from the right, then P_{-1} enters (4), and $P_{-1}(L) = 1$. For $i = 1$, in (4), only the first terms will remain on the right. The amplitudes of the waves diverging in both directions from the irregular section, $P_i(L)$ and $P_{-i}(0)$, are found from (4) by quadratures; the corresponding formulas were obtained in another way in ([4]).

2. If in some section $z = \tilde{z}$, for some i, $h^i(\tilde{z}) = 0$** (a critical section), then S_{im} for all m becomes infinite at $z = \tilde{z}$, and near $z = \tilde{z}$ the S_{im} take large values. Since near a critical section the field cannot be represented as a sum of the fields of forward and backward waves, in order to determine the field in this case one should introduce other variables instead of $P_i(z)$.

Let us introduce***

$$Q_i = P_i - P_{-i}, \qquad R_i = P_i + P_{-i}. \tag{5}$$

According to (^{2}), Q_i and R_i are the expansion coefficients of the transverse components $\mathbf{E}$ and $\mathbf{H}$ in an irregular waveguide in terms of the fields $\mathbf{E}^i$ and $\mathbf{H}^i$ of waves in regular waveguides. The variables Q_i and R_i satisfy second-order equations

$$Q_i'' + (h^i)^2 Q_i = -i \sum_m R_m \{h^i(S_{i,m} + S_{i,-m}) + h^m(S_{i,m} - S_{i,-m})\}(1 - \delta_{i,m}) +$$

$$+ \sum_m Q_m (S_{i,m} - S_{i,-m})' + \sum_m \sum_n Q_n (S_{i,m} - S_{i,-m})(S_{m,n} - S_{m,-n}), \tag{6a}$$

$$R_i'' + (h^i)^2 R_i = -i \sum_m Q_m \{h^i(S_{i,m} - S_{i,-m}) + h^m(S_{i,m} + S_{i,-m})\}(1 - \delta_{i,m}) +$$

$$+ \sum_m R_m (S_{i,m} + S_{i,-m})' + \sum_m \sum_n R_n (S_{i,m} + S_{i,-m})(S_{m,n} + S_{m,-n}), \tag{6b}$$

where the summation over m and n runs from $m, n = 1$ to ∞, and it has been taken into account that $S_{-i,-j} = S_{i,j}$.

According to (2), (3), the coupling coefficients have the property that, if i is a magnetic wave, then the difference $S_{i,m} - S_{i,-m}$ remains finite when $h^i = 0$, while if i is an electric wave, then the sum $S_{i,m} + S_{i,-m}$ remains finite when $h^i = 0$. Therefore, for magnetic waves the right-hand side of (6a) is finite everywhere, including the critical section, and for electric waves the right-hand side of (6b) has no singularities. The first-order equations in ϑ_0, valid everywhere including the neighborhood of a critical section, will be, respectively, for magnetic and electric waves,

$$Q_i'' + (h^i)^2 Q_i = G, \qquad G = -iR_1\{h^i(S_{i,1} + S_{i,-1}) + h^1(S_{i,1} - S_{i,-1})\}(1 - \delta_{i1}); \tag{7a}$$

$$R_i'' + (h^i)^2 R_i = G, \qquad G = -iQ_1\{h^i(S_{i,1} - S_{i,-1}) + h^1(S_{i,1} + S_{i,-1})\}(1 - \delta_{i,1}). \tag{7b}$$

* In this case the ratio of the energy of wave number i to the energy of the incident wave is equal to $|P_i|^2 h^i/h^1$.

** It can always be assumed that the waveguide is narrowing, i.e., that in the critical section $h' < 0$.

*** These same variables are convenient for use in investigating resonance phenomena in waveguides, for example in the problem of a bent waveguide of constant cross section at a frequency coinciding with the critical frequency of the parasitic wave that is produced.

The discarded terms have no singularities in the critical section. Thus, for the general case considered in the present note, the problem reduces to integrating the same equation (7) as in the special case considered in ([1]) (equations (6) and (44)).

3. If $i = 1$, i.e., in the critical section the wave number of the incident wave vanishes, then, to the leading order in ϑ_0, the field is described by a homogeneous equation.

In the critical region, $(h^i)^2$ can be represented by a linear function of $z - \tilde{z}$, and the solution of the homogeneous equation in this region is a linear combination of Airy functions $u(t)$ and $v(t)$ (([5], cf. ([6])) of the argument

$$t = A^{2/3}(z - \tilde{z}), \qquad A^2 = -2\{h^i(h^i)'\}_{z=\tilde{z}}. \tag{8}$$

The derivative $(h^i)'$ is readily computed from (3). For the case considered in ([1]), (8) becomes (([1]), (12)), and all the results of the first three paragraphs of ([1]) remain fully valid when t in ([1]) is replaced by the more general expression (8). For the most interesting special case (to which we restrict ourselves below), when the critical section in the transition waveguide is so far from both its ends that the value of $|t|$ for them is large compared with unity, the phase of the reflected wave differs from twice the "geometrical" phase advance (between $z = 0$ and $z = \tilde{z}$) by $\pi/2$.

4. If $i \neq 1$, then the inhomogeneous equations (7) must be solved. Solving (1) in zeroth order, we obtain, for the quantities entering (7),

$$R_1 = \pm Q_1 = [h^1(0)/h^1]^{1/2} \exp(\mp i\gamma^1), \tag{9}$$

where

$$\gamma^i = \int_0^z h^i(z)\,dz.$$

In (9) the upper sign refers to the case when the wave $i = 0$ is incident on the transition from the left, and the lower sign to the case when it is incident from the right; in the latter case $h^1(0)$ must be replaced by $h^1(L)\exp[-2i\gamma^1(L)]$.

Equations (7) are not completely equivalent to (4); in deriving them it was essentially assumed that ϑ' is small everywhere, i.e., bends of the waveguide surface were excluded. The application of second-order equations to long, smoothly varying waveguides also leads, in some cases, to difficulties. Therefore we shall use equations (7) only near the critical section, and far from it we shall apply (4). With the aid of (7) one can establish the relation between $P_i(z)$ and $P_{-i}(z)$ for z lying in the region of applicability of (4). This relation makes it possible to find $P_{-i}(0)$—the sought amplitude of the wave leaving into the wide waveguide —from (4) and the condition $P_i(0) = 0$, which ensures the absence of a wave of number i incident on the irregular section. Thus, describing the field in the critical region by means of (7), one can establish for (4) an end condition equivalent to the presence of a critical section, and subsequently exclude the critical region from consideration. In [1] this method is applied to the case $i = 1$.

The equivalent end condition is obtained from the general solution of equations (7) and the requirement that the field of wave number i decrease with increasing z for $z > \tilde{z}$. This solution is expressed in terms of a function $V(z)$ satisfying the homogeneous equation $V'' + (h^i)^2 V = 0$ and decreasing for $z > \tilde{z}$. In the critical region, for $|t| \lesssim 1$, $V(z)$ is proportional to $v(t)$ (10a); for $|t| \gg 1$, $V(z)$ goes over into the solution (10), valid in the geometrical-optics approximation:

$$V(z) = A^{-1/3} v(t); \tag{10a}$$

$$V(z) = (h^i)^{-1/2} \sin(-\gamma^i + \tilde{\gamma}^i + \pi/4), \qquad (10\ )$$

where $\tilde{\gamma}^i = \gamma^i(\tilde{z})$ (cf. (1)). For magnetic waves, if the exciting wave is incident from the left, the end condition has the form

$$P_i(z)\exp[i(\gamma^i - \tilde{\gamma}^i - \pi/4)] - P_{-i}(z)\exp[-i(\gamma^i - \tilde{\gamma}^i - \pi/4)] = \left[h^1(0)/h^i\right]^{1/2} \int_z^L (h^1)^{-1/2} \exp(-i\gamma^1)\{S_{i1}[ih^{iV} + V'] + S \tag{11}$$

In an analogous manner it is written for electric waves and for incidence of the wave $i = 1$ from the right.

According to (4) and the condition $P_i(0) = 0$, for any z outside the critical region (and for $z < \tilde{z}$),

$$P_i(z) = \left[h^1(0)/h^i\right]^{1/2} \exp(-i\gamma^i) \int_0^z (h^1)^{-1/2} \exp(-i\gamma^1) S_{i,1} \left[(h^i)^{1/2} \exp(i\gamma^i)\right] dz; \tag{12}$$

$$P_{-i}(z) = \tag{13}$$

$$= \left[h^1(0)/h^i\right]^{1/2} \exp(i\gamma^i) \int_0^z (h^1)^{-1/2} \exp(-i\gamma^1) S_{i,-1} \left[(h^i)^{1/2} \exp(-i\gamma^i)\right] \, dz + P_{-i}(0).$$

From equations (11)–(13) the required amplitude $P_{-i}(0)$ is determined. The value of z in them may be arbitrary (outside the critical region), since the end condition (11) is written in such a form that, for large $|z - \tilde{z}|$, the factors at $S_{i,1}$ and $S_{i,-1}$ in (11) pass, according to (10), with accuracy up to $\exp[\pm i(\tilde{\gamma}^i + \pi/4)]$, into the corresponding factors in (12) and (13).*

Outside the critical region, the forward and backward waves are formed independently of each other (equations (12) and (13)). In the critical region, the entire parasitic field is formed as a whole (equation (11)). The use of the end condition (11) ensures a uniform analysis of the field throughout the irregular waveguide.

Let us note, in conclusion, that the frequency dependence of the integral expressions in (11)–(13) is mainly determined, according to (2), by the factors $\exp[i(-\gamma^1 \pm \gamma^i)]$. Introducing, instead of z, the variable $-\gamma^1 + \gamma^i$, one can reduce the calculation of the amplitude of a parasitic wave in irregular waveguides to the calculation of the reflection coefficient from an irregular long line. This makes it possible to apply to the design of waveguide transitions with prescribed characteristics the known methodology (see, for example, (7)); one can also use the results of works on antennas, for example (8), for determining the optimal parameters of long lines.

Institute of Radio Engineering and Electronics
Academy of Sciences of the USSR

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* For not very large $|z - \tilde{z}|$, the end condition may be written in various forms equivalent to one another. The end condition written in the form (12) may be applied for any z, including $z = 0$. In this case it gives $P_i(0)$ in the form of a single integral. The integral expression in the critical region is calculated according to (11a), outside the critical region according to (11); the result obtained is, naturally, the same as when using (13) and (14). In other words, the solution of equations (7a) in the form (12) is equivalent to the solution of the first-order equations (4).

Note: Figure translations are in progress. See original paper for figures.