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Abstract

Full Text

MATHEMATICS

B. M. LEVITAN

ADJOINT OPERATORS OF GENERALIZED TRANSLATION

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1. Let V_n be a real differentiable manifold and let T^s be a family of generalized translation operators defined on V_n (see ([1])). Let $m(E)$ be a completely additive measure on V_n. We define the family of adjoint operators $\widetilde{T}^s$ by means of the equality

$$\int_{V_n} T_t^s f(t)\,\overline{g(t)}\,dm(t) = \int_{V_n} f(t)\,\overline{\widetilde{T}_t^s g(t)}\,dm(t). \tag{1}$$

In ([2]) we studied properties of adjoint operators. Some of these properties do not depend on the measure m, but the deepest properties of the adjoint operators depend essentially on the measure.

We require that the adjoint operators satisfy the conditions:

a) in the noncommutative case

$$\widetilde{T}_s^r T_t^s f(t) = T_t^s \widetilde{T}_t^r f(t); \tag{2}$$

b) in the commutative case

$$\widetilde{T}_t^s T_t^r f(t) = T_t^r \widetilde{T}_t^s f(t). \tag{3}$$

It can be shown that in the commutative case, for $f(t)$ from the subspace M (see condition 3° of generalized translation in ([1])), conditions (2) and (3) are equivalent. Therefore, in what follows we shall confine ourselves to the study of property (2).

In this note we shall consider case II (see ([1])). Thus, $u(s,t) = T_t^s f(t)$ is the solution of the Cauchy problem

$$\widetilde{N}_{\alpha;s} u = N_{\alpha;t} u; \tag{4}$$

$$u\Big|_{s=0} = f(t); \tag{5}$$

$$\left.\frac{\partial^\lambda u}{\partial s_1^{\lambda_1} \cdots \partial s_n^{\lambda_n}}\right|_{s=0} = h_{\lambda_1,\dots,\lambda_n} f(t), \tag{6}$$

where $\widetilde{N}_{\alpha;s}$ and $N_{\alpha;t}$ $(\alpha = 1, 2, \dots, n)$ are differential operators of the second order.

Theorem 1. Denote by $N^*_{\alpha;t}$ the operators adjoint to the operators $N_{\alpha;t}$ with respect to the measure m.

The function $v(s,t) = \widetilde{T}^s_t g(t)$ is the unique solution of the following Cauchy problem

$$\overline{N}_{\alpha;s} v = N^*_{\alpha;t} v; \tag{7}$$

$$v\Big|_{s=0} = g(t); \tag{8}$$

$$\left.\frac{\partial^\lambda v}{\partial s_1^{\lambda_1} \dots \partial s_n^{\lambda_n}}\right|_{s=0} = \overline{h}_{\lambda_1,\dots,\lambda_n} g(t), \tag{9}$$

where $\overline{N}_{\alpha;s}$ are operators with complex-conjugate coefficients.

Proof. Denote by $h(s)$ the left-hand side (which has also become the right-hand side) of equality (1), and apply the operator $N_{\alpha;s}$ to $h(s)$. We obtain (using the commutation condition $N_{\alpha;s} T^s_t f(t) = T^s_t N_{\alpha;t} f(t)$)

$$N_{\alpha;s} h(s) = \int_{V_n} T^s_t N_{\alpha;t} f(t)\, \overline{g(t)}\, dm(t) = \int_{V_n} f(t)\, \overline{N^*_{\alpha;t} \widetilde{T}^s_t g(t)}\, dm(t) =$$

$$= \int_{V_n} f(t)\, \overline{N_{\alpha;s} \widetilde{T}^s_t g(t)}\, dm(t).$$

Hence, and from the arbitrariness of the function $f(t)$, equation (7) follows. Setting $s = 0$ in (1), we obtain condition (8). Differentiating (1) λ_1 times with respect to s_1, λ_2 times with respect to s_2, etc., λ_n times with respect to s_n, and then setting $s = 0$, we obtain condition (9).

If the operators T^s transform real functions into real ones, then the coefficients of the operators $\widetilde{N}_{\alpha;s}$ and $N_{\alpha;t}$, and the numbers $h_{\lambda_1,\dots,\lambda_n}$, are real. In this case the system (7), (8), (9) takes the form

$$N_{\alpha;s} v = N^*_{\alpha;t} v; \tag{7'}$$

$$v\Big|_{s=0} = g(t); \tag{8'}$$

$$\frac{\partial^\lambda v}{\partial s_1^{\lambda_1} \dots \partial s_n^{\lambda_n}}\Bigg|_{s=0} = h_{\lambda_1,\dots,\lambda_n} g(t). \tag{9'}$$

In what follows we shall consider the real case.

2. Let

$$N_{\alpha;t}(f) = a^{ij}_\alpha(t)\frac{\partial^2 f}{\partial t_i \partial t_j} + b^i_\alpha(t)\frac{\partial f}{\partial t_i} + c_\alpha(t) f$$

with real coefficients, and suppose that the measure $m(E)$ in local coordinates can be represented in the form

$$m(E) = \int_E a(t)\, dt_1 \dots dt_n,$$

where $a(t)$ is a real differentiable function. It is not difficult to verify that in this case the adjoint operators $N^*_{\alpha;t}$ have the form

$$N^*_{\alpha;t}(g) = \frac{1}{a(t)}\left[\frac{\partial^2}{\partial t_i \partial t_j}(a a^{ij}_\alpha g) - \frac{\partial}{\partial t_i}(a b^i_\alpha g) + c_\alpha a g\right]. \tag{10}$$

Theorem 2. In order that the adjoint operators satisfy condition (2), it is necessary and sufficient that, for an arbitrary function $f(t)$, the condition

$$N^*_{\alpha;s} T^s_t f(t) = T^s_t N^*_{\alpha;t} f(t). \tag{11}$$

be fulfilled.

Proof. Since

$$N_{\alpha;t}(f) = \frac{\partial^2 T^s_t f(t)}{\partial s^2_\alpha}\Bigg|_{s=0}, \qquad N^*_{\alpha;t}(f) = \frac{\partial^2 \widetilde{T}^s_t f(t)}{\partial s^2_\alpha}\Bigg|_{s=0},$$

(11) follows from (2).

To prove that (2) follows from (11), introduce the notation

$$\Phi(s,r,t) = \widetilde{T}_s^r T_t^s f(t), \qquad \Psi(s,r,t) = T_t^s \widetilde{T}_t^r f(t).$$

The function $\Phi(s,r,t)$, as a function of the variables s and r, is the solution of the following Cauchy problem:

$$N_{\alpha;r}\Phi = N^*_{\alpha;s}\Phi; \tag{12}$$

$$\Phi\big|_{r=0} = T_t^s f(t); \tag{13}$$

$$\left.\frac{\partial^\lambda \Phi}{\partial r_1^{\lambda_1} \dots \partial r_n^{\lambda_n}}\right|_{r=0} = h_{\lambda_1,\dots,\lambda_n} T_t^s f(t). \tag{14}$$

We shall show that Ψ also satisfies these conditions.

Conditions (13) and (14) are verified directly. Let us verify equations (12). We have, using (7′) and (11):

$$N_{\alpha;r}\Psi = T_t^s N_{\alpha;r}\widetilde{T}_t^r f(t) = T_t^s N^*_{\alpha;t}\widetilde{T}_t^r f(t) = N^*_{\alpha;s}\Psi.$$

Remark. Differentiating equality (11) twice with respect to t_β and then putting $t = 0$, we obtain

$$\widetilde{N}_{\beta;s} N^*_{\alpha;s}(f) = N^*_{\alpha;s}\widetilde{N}_{\beta;s}(f). \tag{15}$$

Thus condition (15) is a consequence of condition (11). It can be shown that, for the equivalence of conditions (11) and (15) (and consequently also of conditions (2) and (15)), it is necessary and sufficient that, for any function $f(t)$, the following hold:

$$\begin{aligned} &1)\quad N^*_{\alpha;s} T_t^s f(t)\big|_{s=0} = N^*_{\alpha;t} f(t);\\ &2)\quad \left.\frac{\partial^\lambda}{\partial s_1^{\lambda_1} \dots \partial s_n^{\lambda_n}} N^*_{\alpha;s} T_t^s f(t)\right|_{s=0} = h_{\lambda_1,\dots,\lambda_n} N^*_{\alpha;t} f(t). \end{aligned} \tag{16}$$

3. The situation is greatly simplified in the case when $N_{\alpha;t}$ are first-order operators, which corresponds to a shift on a group.

Thus, let

$$N_{\alpha;t}(f) = b_\alpha^i(t)\frac{\partial f}{\partial t_i}, \qquad \widetilde{N}_{\alpha;s}(f) = \tilde{b}_\alpha^i(s)\frac{\partial f}{\partial s_i},$$

where

$$b_\alpha^i(0) = \tilde{b}_\alpha^i(0) = \delta_\alpha^i.$$

In this case the adjoint operators have the form

$$N_\alpha^*(g) = -\,N_\alpha(g) - \frac{1}{a}N_\alpha(a)\,g - B_\alpha g, \tag{17}$$

where $B_\alpha(t) = \partial b_\alpha^i(t)/\partial t_i$ (summation over i).

Further, if $u(s,t) = T_t^s f(t)$, then

$$u\big|_{s=0} = f(t), \qquad N_{\alpha;s}(u)\big|_{s=0} = N_{\alpha;t}(f).$$

Therefore from condition (16) it follows that

$$\frac{1}{a}N'_\alpha(a) + B\alpha = \lambda_\alpha, \tag{18}$$

where λ_α are constant numbers. Each solution of system (17) gives a measure $a(t)$ for which the corresponding adjoint operators satisfy condition (3). From (17) and (18) it follows that the adjoint operators have the form

$$N_\alpha^*(g) = -N_\alpha(g) - \lambda_\alpha g.$$

For $\lambda_\alpha = 0$, $a(t)$ satisfies the system

$$\frac{1}{a}N_\alpha(a) + B\alpha = 0$$

and is the density of the Haar measure.

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[1] B. M. Levitan, DAN, **123**, No. 1 (1958). [2] B. M. Levitan, Uspekhi matem. nauk, **6**, issue 1, 3 (1949).

Note: Figure translations are in progress. See original paper for figures.