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Abstract

Full Text

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ON THE ASYMPTOTIC REPRESENTATION OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

(Presented by Academician A. A. Dorodnitsyn, 27 VI 1957)

1. Consider the differential equation

$$k(x)y'' + p(x)y' + [\lambda^2 \rho(x) + q(x)]y = 0, \qquad a \le x \le b. \tag{1}$$

The following scheme generalizes various methods ($^{1-13}$) for obtaining asymptotic representations of solutions of equation (1) for large λ, both in the case when the equation has no singular points and in the case when they are present.

By a substitution of the form

$$x = x(t), \qquad y = A(t)z, \qquad c \le t \le d, \tag{2}$$

equation (1), on some interval $a \le a_1 \le x \le b_1 \le b$, is reduced to a form not containing the first-derivative term:

$$z'' + [\lambda^2 \varphi(t) + \psi(t)]z = 0. \tag{3}$$

Then, in the general case, a "nearby" equation is chosen for this equation,

$$U'' + [\lambda^2 \varphi(t) + \psi(t) + \omega(t)]U = 0, \tag{4}$$

in the sense that the function $\omega(t)$ is everywhere on $[c, d]$, for sufficiently large λ, small in comparison with $\lambda^2 \varphi(t) + \psi(t)$.

The function $\omega(t)$ is chosen so that, by the substitution

$$\xi = \xi(t), \qquad V = \sqrt{\frac{d\xi}{dt}}\, U \tag{5}$$

equation (4) is reduced to the equation

$$V'' + \sigma(\xi)V = 0, \qquad -\infty \le \gamma \le \xi \le \delta \le +\infty, \tag{6}$$

whose general solution is written in closed form in terms of known functions $V_1(\xi)$ and $V_2(\xi)$:

$$V(\xi) = c_1V_1(\xi) + c_2V_2(\xi), \qquad W\{V_1, V_2\} \neq 0.$$

Returning to the variables t and U, we have

$$U(t) = \sqrt{\frac{dt}{d\xi}}\,[c_1V_1(\xi(t)) + c_2V_2(\xi(t))].$$

The general solution of equation (3) is sought in the form

$$z(t) = \sqrt{\frac{dt}{d\xi}}\,[c_1(t)V_1(\xi(t)) + c_2(t)V_2(\xi(t))].$$

Applying the method of variation, we obtain, for determining $z(t)$, a Volterra integral equation, whose solution is usually found by the method of successive approximations. The series thus obtained for determining $z(t)$ is at the same time an asymptotic series in λ. It remains only to make the substitution (2) in order to obtain an asymptotic representation of the solution $y(x)$ of equation (1).

In the choice of the function $\xi(t)$ in the substitution (5) there is a certain arbitrariness; however, all possible cases are subject to the following condition (([14], § 64, pp. 115-117):

The solutions of equations (4) and (6) are related to each other by the relation $V(\xi) = \sqrt{\frac{d\xi}{dt}}\,U(t)$ if and only if $\xi = \xi(t)$ satisfies the differential equation

$$\left(\frac{d\xi}{dt}\right)^2 \sigma(\xi) = \lambda^2\varphi(t) + \psi(t) + \omega(t) - \frac{1}{2}\{\xi, t\}, \tag{7}$$

where $\{\xi, t\}$ is the Schwarzian derivative.

Choosing the function $\omega(t)$ in (7) in various ways, we each time obtain a quite definite function $\xi(t)$, whose character, of course, will depend primarily on the form of the function $\sigma(\xi)$.

A more general device for finding the function $\xi(t)$, it seems to us, should be considered the following.

Putting

$$\omega(t) = \frac{1}{2}\{\xi, t\} - \psi(t) + \left(\frac{d\xi}{dt}\right)^2 s(\xi), \tag{8}$$

we obtain, for determining $\xi(t)$, the equation

$$\frac{d\xi}{dt} = \lambda\sqrt{\frac{\varphi(t)}{\sigma(\xi) - s(\xi)}},$$

where the function $s(\xi)$ is chosen from considerations of simplicity of the computations. Having determined $\xi(t)$ in this way, we thereby determine $\omega(t)$ definitively.

Remarks. 1) The functions participating in the change of variables may have singularities, which must be strictly taken into account in all subsequent computations.

2) The form of the substitution (5) is dictated by the desire to eliminate the possibility of the appearance of a term with the first derivative and to obtain in the left-hand side the required operator $L[V] \equiv V'' + \sigma(\xi)V$, $\gamma \leq \xi \leq \delta$.

3) The first terms of the asymptotic representations will contain one or another set of well-studied functions (trigonometric, Bessel, etc.), depending on which operator $L[V]$ is written on the left-hand side of equation (6). The form of this operator is determined from equation (3) (([12], p. 4).

4) To obtain asymptotic representations of the solutions $y(x)$ of equation (1) on the whole interval $[a, b]$ in the presence of two or more singular points, it is also necessary to join together the separate pieces of the solutions $y(x)$. The least number and the arrangement of the intervals $[a_1, b_1]$ on which the constructions described above are carried out are determined by the number and position on $[a, b]$ of the singular points of equation (1).

2. Let us apply this scheme to the construction of asymptotic representations on the whole interval of solutions of an equation containing a singularity of a type more general than that considered earlier:

$$x^\alpha y'' + x^{\alpha-1}p(x)y' + \left[\lambda^2 x^\alpha + \beta r(x) - q(x)\right] y = 0, \qquad 0 \leq x \leq l, \tag{9}$$

where $p(x)$ is a differentiable function, $0 \leq \alpha \leq 2$, $\beta > -2$, $x^{2-\alpha}q(x) \in \mathrm{Lip}\,1$, and $r(x) > 0$ is a twice differentiable function.

Making the substitution

$$y(x) = x^{-p(0)/2} \exp\left[-\frac{1}{2}\int_0^x \frac{p(x) - p(0)}{x}\,dx\right] z(x), \tag{10}$$

we obtain

$$z'' + \left[\lambda^2 x^\beta r(x) + s x^{-2} + q_1(x) x^{-1}\right] z = 0, \tag{11}$$

where

$$q_1(x) = -Q_1(x) + \frac{p(0)p_1(x)}{2} + \frac{x p_1^2(x)}{4} - \frac{x p_1'(x)}{2};$$

$$x Q_1(x) = -Q(x) - Q(0); \qquad x p_1(x) = p(x) - p(0);$$

$$s = \frac{p(0)}{2}\left(1 - \frac{p(0)}{2}\right) - Q(0);$$

$$Q(x) = \begin{cases} q(x), & \alpha = 2, \\ q(x) x^{2-\alpha}, & \alpha < 2. \end{cases}$$

Immediately, from equation (11), we choose the operator

$$L[V] = V'' + \left[\xi^\beta + s\xi^{-2}\right] V, \qquad 0 \leqslant \xi \leqslant \infty.$$

Consequently, we define the function $\xi = \xi(t)$ from the equation

$$\frac{d\xi}{dt} = \lambda \sqrt{\frac{x^\beta r(x)}{\xi^\beta + s\xi^{-2} - s(\xi)}}.$$

If one chooses $s(\xi) = s\xi^{-2}$, then it turns out that the equation

$$U'' + \left[\lambda^2 x^\beta r(x) + s\xi^{-2} + q_1(x) x^{-1} + \omega(x)\right] U = 0, \tag{12}$$

where

$$\omega(x) = -\sqrt{\frac{dH}{dx}}\frac{d^2}{dx^2}\left[\left(\frac{dH}{dx}\right)^{-1/2}\right] + S\left[\frac{x^\beta r(x)}{H^{\beta+2}(x)} - \frac{1}{x}\right] - \frac{q_1(x)}{x};$$

$$H(x) = \left[\frac{\beta+2}{2}\int_0^x \sqrt{x^\beta r(x)}\, dx\right]^{2/(\beta+2)},$$

by the substitution

$$\xi = \lambda^{2/(\beta+2)} H(x), \qquad V = \lambda^{1/(\beta+2)} \sqrt{\frac{dH}{dx}}\, U \tag{13}$$

is transformed into the equation

$$V'' + \left[\xi^{\beta} + s\xi^{-2}\right] V = 0. \tag{14}$$

Note that $\omega(x) = O\left(\frac{1}{x}\right)$, whereas

$$\lambda^2 x^{\beta} r(x) + s x^{-2} + q_1(x) x^{-1} = O\left(\frac{1}{x^2}\right), \qquad x \to 0.$$

Representing equation (11) in the form

$$z'' + \left[\lambda^2 x^{\beta} r(x) + s x^{-2} + q_1(x) x^{-1} + \omega(x)\right] z = \omega(x) z,$$

we seek its solution in the form

$$z(x) = c_1(x) U_1(x) + c_2(x) U_2(x),$$

$U_{1,2}(x)$ are two linearly independent solutions of equation (12). According to (13),

$$U_{1,2}(x) = \lambda^{-1/(\beta+2)} \left(\frac{dH}{dx}\right)^{-1/2} V_{1,2}(\xi(x)),$$

where $V_{1,2}(\xi)$ are two linearly independent solutions of equation (14), which, in turn, are expressed in terms of Bessel functions of arbitrary index. This may be verified by making in equation (14) the substitution

$$\zeta = \frac{2}{\beta+2} \xi^{(\beta+2)/2}, \qquad J = \left(\frac{\beta+2}{2}\zeta\right)^{\beta/2(\beta+2)} V.$$

Thus, finally we take

$$U_{1,2}(x) = \left(\frac{2}{\beta+2}\right)^{1/2} \left(\frac{H(x)}{dH(x)/dx}\right)^{1/2} J_{\pm\nu}\left(\lambda \int_0^x \sqrt{x^{\beta} r(x)}\, dx\right), \tag{15}$$

where

$$\nu = \frac{\sqrt{1 - p(0)[2 - p(0)] + 4Q(0)}}{\beta+2}, \qquad \nu \neq 0, 1, 2, \ldots.$$

In the case $\nu = 0, 1, 2, ...$, one of the solutions $U_{1,2}(x)$ must be replaced in the appropriate way.

Using (15), in the most general case of equation (9) one can obtain a representation only for one particular solution of equation (11), from which one solution of equation (9) is obtained in the form

$$y_1(x,\lambda) = \left(\frac{2}{\beta+2}\right)^{1/2} x^{-p(0)/2} \left(\frac{H(x)}{dH(x)/dx}\right)^{1/2} \times$$

$$\times \exp\left[-\frac{1}{2}\int_0^x \frac{p(x)-p(0)}{x}\,dx\right] J_\nu\left(\lambda\int_0^x \sqrt{x^\beta r(x)}\,dx\right) + O\left(\lambda^{-1} x^{-p(0)/2} \max_{0\le t\le x} U_1(t)\right). \tag{16}$$

Under additional assumptions on the functions $p(x)$ and $q(x)$, further terms of the asymptotic representation (16) may also be computed.

The solution (16) is regular at $x = 0$. The second solution of equation (9), linearly independent of (16), is found by the usual methods for linear second-order equations.

In special cases of equation (9) ((*12*), pp. 36, 63), the scheme makes it possible to obtain a representation immediately for two linearly independent solutions. In these cases, however, more stringent restrictions are imposed on $r(x)$ and $q(x)$.

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