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Abstract

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APPLICATION OF THE THEORY OF SEMIGROUPS TO THE STUDY OF DIFFERENTIAL EQUATIONS IN BANACH SPACES

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The article considers the equation

$$\frac{du}{dt} + Au = f(t) \tag{1}$$

with the initial condition

$$u\big|_{t=0} = u_0, \tag{2}$$

where $u = u(t)$ and $f(t)$ are elements of a complex Banach space $\mathfrak{X}$, depending on the parameter $t \in [0, T]$; A is a closed unbounded operator with domain of definition $D(A)$ dense in $\mathfrak{X}$; equations of a somewhat more general form are also studied.

If the operator $-A$ is the infinitesimal generator of some semigroup of bounded linear operators $T(\xi)$ (1), then the formal solution of problem (1)–(2) can be written in the form

$$u(t) = T(t)u_0 + \int_0^t T(t-\tau)f(\tau)\,d\tau. \tag{3}$$

However, the question arises: in what sense does (3) satisfy the equation and the initial data; in particular, under what conditions does the function $u(t)$, defined by formula (3), have a strong derivative and belong to $D(A)$ for all (or, at least, for almost all) values of t.

Under minimal assumptions concerning the operator A, this question was investigated by Phillips (2), who had to impose very stringent restrictions on the right-hand side and the initial condition, namely: $u_0 \in D(A)$, $f(t)$ is strongly continuously differentiable. These restrictive conditions have to be imposed because in (2) the properties of the resolvent $R(\lambda; A) = (\lambda I - A)^{-1}$ of the operator A are used only on the negative real half-axis. More detailed information about the spectral properties of the operator A makes it possible to reduce substantially the requirements on u_0 and $f(t)$. In essence, the results of the work of M. A. Krasnosel' skii, S. G. Krein, and P. E. Sobolevskii (^{3}), in which the question is studied in Hilbert space, are based on this. Here all the results obtained for problem (1)—(2) are based on the estimate

$$\|A^{\alpha}T(\xi)\| \leq \frac{C(\alpha)}{\xi^{\alpha}} \qquad (\alpha > 0) \tag{4}$$

(the operator A is assumed to be self-adjoint and positive-definite).

We shall show that, under certain conditions, an analogous estimate can also be obtained for operators acting in a Banach space.

Theorem 1. Let the resolvent set of the operator A contain a certain sector Σ: $\varphi \leq \arg\lambda \leq 2\pi - \varphi$ ($\varphi < \pi/2$), and suppose that for $\lambda \in \Sigma$ the inequality

$$\|R(\lambda; A)\| \leq \frac{C}{|\lambda| + 1} \tag{5}$$

holds. Then the formula

$$T(\xi)x = \frac{1}{2\pi i}\int_{\Gamma} e^{-\lambda\xi}R(\lambda; A)x\,d\lambda, * \tag{6}$$

where Γ is the boundary of the sector Σ, defines a semigroup $T(\xi)$, whose infinitesimal generator is $-A$, and:

1) As $\xi \to 0$, $T(\xi)$ tends strongly to I.
2) $T(\xi)$ is holomorphic in the sector $-(\pi/2 - \varphi) < \arg\xi < \pi/2 - \varphi$.
3) The operators $A^{nT}(\xi)$ are bounded and are represented by the integrals

$$A^{nT}(\xi)x = \frac{1}{2\pi i}\int_{\Gamma} \lambda^{n} e^{-\lambda\xi}R(\lambda; A)x\,d\lambda \tag{7}$$

and for them the estimate

$$\|A^{nT}(\xi)\| \leq \frac{C_n}{\xi^{n}} \quad (n \geq 0) \tag{8}$$

holds

(ξ real positive).

This theorem is close to Theorem 12.8.1 of the book (^{4}) and can easily be obtained from it.

Remark. The circumstance that the vertex of the sector Σ is located at the origin is inessential. The general case is reduced to the one considered by replacing $A_1 = A + kI$; in equation (1) this corresponds to the substitution $u(t) = e^{kt}v(t)$.

Theorem 1 makes it possible to extend to Banach spaces certain results analogous to those obtained in paper (^{3}) for Hilbert spaces. Everywhere in what follows we shall assume that the operators A under consideration satisfy the conditions of Theorem 1.

By a **classical solution** of problem (1)–(2) we shall mean a function $u(t)$ which, for all $t \in (0, T]$, is strongly continuously differentiable, belongs to $D(A)$, and satisfies equation (1), and which, as $t \to 0$, tends strongly to u_0. In addition, we shall consider generalized solutions of various classes. We shall call a function $u(t)$ a **generalized solution** of problem (1)–(2) of class $\mathcal{B}_p$ if, for $t > 0$, $u(t)$ is absolutely continuous, is strongly differentiable almost everywhere and belongs to $D(A)$, and if du/dt and Au belong to $\mathcal{B}_p$ (the Bochner class) on the interval $[0, T]$. The equation must be satisfied almost everywhere; satisfaction of the initial condition is still understood in the sense of strong convergence. If it is known only that $\|du/dt\|^p$ is summable on any interval $[\varepsilon, T]$ ($\varepsilon > 0$), then we shall say that $u(t)$ is a generalized solution of class $\mathcal{B}'_p$. Let us note that a classical solution, if it exists, is also a generalized solution of class $\mathcal{B}'_p$, but not always of class $\mathcal{B}_p$.

The uniqueness of the solution of problem (1)–(2) was proved in (^{4}), Ch. XXIII; the proof is readily extended also to the case of generalized solutions.

Theorem 2. Let $u_0 \in \mathfrak{X}$; let $f(t)$ satisfy the condition $\operatorname{Lip}\varepsilon$, $\varepsilon \leq 1$. Then formula (3) defines a classical solution of problem (1)–(2), and $du/dt \in \operatorname{Lip}\delta$ for arbitrary $0 < \delta < \varepsilon$.

* The contour is traversed from bottom to top.

Theorem 3. Let $u_0 \in \mathfrak{X}$; $f(t) \in \mathfrak{B}_p$ ($p > 1$) and, as a function from $\mathfrak{B}_p$, satisfy the condition $\operatorname{Lip}\varepsilon$ ($\varepsilon \leqslant 1$); the latter means that

$$\left\{ \int_0^{T-h} \|f(t+h) - f(t)\|^p \, dt \right\}^{1/p} \leqslant Ch^{\varepsilon}.$$

Then formula (3) defines a generalized solution of problem (1)–(2) of class $\mathfrak{B}'_p$.

A number of results can be obtained if one brings into consideration fractional powers of the operator A. In (1), Chap. XV, a theory is presented which makes it possible, for operators satisfying the conditions of Theorem 1, to construct negative powers $A^{-\beta}$; after this, fractional positive powers can be defined, for example, by the formula

$$A^{\alpha} x = A^{n}[A^{-(n-\alpha)} x],$$

where $n = [\alpha] + 1$. In this case the operators $A^\alpha T(\xi)$ are bounded and are represented by the integrals

$$A^\alpha T(\xi)x = \frac{1}{2\pi i}\int_\Gamma \lambda^\alpha e^{-\lambda\xi} R(\lambda; A)x\, d\lambda, \tag{9}$$

$$\|A^\alpha T(\xi)\| \leqslant \frac{C(\alpha)}{\xi^\alpha} \quad (\alpha \geqslant 0) \tag{10}$$

(in the integral (9) the values of λ with the smallest argument in absolute value are taken).

Theorem 4. If $u_0 \in D(A^\alpha)$, where $\alpha > 1 - 1/p$, then under the conditions of Theorem (3) $u(t)$ is a generalized solution of class $\mathfrak{B}'_p$.

Theorem 5. If $f(t) \in \mathfrak{B}_p$, then for $t > 0$ formula (3) defines a function $u(t) \in D(A^\alpha)$ $(\alpha < 1 - 1/p)$, and the function $A^\alpha u(t)$ is strongly continuous for $t > 0$.

Let us pass to the study of the equation

$$\frac{du}{dt} + Au + B(t)u = f(t), \tag{11}$$

where $B(t)$ is an operator in some sense subordinate to A. Such equations (with bounded $B(t)$) were considered by Phillips (^{2}). The case when the operator A itself may depend on t was studied by Kato (^{5},^{6}) under different assumptions on the properties of A. In the case of a Hilbert space, in (^{3}), for operators satisfying Kato' s conditions, estimates generalizing (^{4}) were obtained. However, we have not succeeded in obtaining analogous results in the case of an arbitrary Banach space.

Theorem 6. Let $u_0 \in \mathfrak{X}$, let $f(t)$ satisfy the conditions of Theorem 2, and suppose that there exist constants α and δ $(\alpha > 0,\ \delta > 0,\ \alpha + \delta < 1)$ such that the inequalities

$$\|A^{-\alpha} B(t)\| \leqslant M, \qquad \|A^\delta B(t) A^{-(\alpha+\delta)}\| \leqslant M,$$

hold, and moreover the operator function $A^\delta B(t) A^{-(\alpha+\delta)}$ is uniformly continuous in t. Then the classical solution of problem (11)–(2) exists, is unique, and can be determined by the method of successive approximations, if $B(t)u$ is regarded as a perturbation.

Also of interest is the case when the operator B does not depend on t and is connected with A by the following relation: for any $\varepsilon > 0$ there exists a number $K(\varepsilon)$ such that for all $u \in D(A)$ the inequality

$$\|Bu\| \leqslant \varepsilon\|Au\| + K(\varepsilon)\|u\| \tag{12}$$

holds.

Theorem 7. If the operator A satisfies the conditions of Theorem 1, and B satisfies condition (12), then all points λ of the sector Σ lying outside a certain circle with center at the origin belong to the resolvent set of the operator $A+B$, and at these points the inequality

$$\|R(\lambda; A+B)\| \leq \frac{C}{|\lambda|}$$

holds.

This theorem shows that all the results proved for equations with the operator A extend to equations with the operator $A+B$.

The results obtained can be applied to the study of equations of parabolic type in the spaces L_p. For this it is necessary to establish that the elliptic operator entering the equation satisfies the conditions of Theorem 1.

Let Ω be a finite domain of n-dimensional Euclidean space with twice continuously differentiable boundary S. In the domain Ω consider the equation

$$\lambda u - Au = f(x) \tag{13}$$

with the boundary condition

$$u\big|_S = 0. \tag{14}$$

Here

$$Au = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(x)\frac{\partial u}{\partial x_k} + c(x)u$$

is a nondegenerate differential expression of elliptic type; the coefficients $a_{ij}(x)$ are assumed to be continuously differentiable, and $b_k(x)$ and $c(x)$ bounded.

Theorem 8. Whatever sector Σ of the complex plane not containing the positive real half-axis may be, for all points $\lambda \in \Sigma$ lying outside a certain circle with center at the origin, the problem (13)–(14), for any $f(x) \in L_p(\Omega)$, has a generalized solution $u(x) \in W_p^2(\Omega)$, and for it the estimate

$$\|u\|_{L_p} \leq \frac{C}{|\lambda|}\|f\|_{L_p}$$

holds.

The method of proof is analogous to that used by A. I. Koshelev in [7].

We also note that if A is a strongly elliptic operator of order $2m$, whose leading coefficients are m times continuously differentiable, then under the boundary conditions

$$u\Big|_S = \frac{\partial u}{\partial n}\Big|_S = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}}\Big|_S = 0,$$

its spectrum is contained in some sector with opening $< \pi$, and for points λ situated outside this sector the inequality (5) holds. We have succeeded in proving this estimate only for the space L_2, but apparently it is also true in any L_p $(p > 1)$.

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Note: Figure translations are in progress. See original paper for figures.

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