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Abstract

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MATHEMATICS

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ON MIXED PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS WITH TWO INDEPENDENT VARIABLES

Consider an equation with constant coefficients

$$\frac{\partial^n u}{\partial t^n} + \sum_{k<n,\, l\le m} A_{k,l} \frac{\partial^{k+l} u}{\partial t^k \partial x^l} = f \tag{1}$$

with two independent variables and one unknown function, satisfying Petrovskii' s condition of uniform correctness of the Cauchy problem. The structure of this equation was analyzed by us in the preceding note ([1]). We shall study this equation in the domains:

$$\begin{aligned} &\text{a) } -\infty < x < +\infty, \quad 0 \le t < +\infty; \\ &\text{b) } 0 \le x < +\infty, \quad 0 \le t < +\infty; \\ &\text{c) } 0 \le x \le 1, \quad 0 \le t < +\infty. \end{aligned} \tag{2}$$

under the initial conditions

$$u\Big|_{t=0} = \frac{\partial u}{\partial t}\Big|_{t=0} = \cdots = \frac{\partial^{n-1} u}{\partial t^{n-1}}\Big|_{t=0} = 0 \tag{3}$$

and under certain homogeneous conditions on the boundaries:

$$\sum_{i=1}^{m-1} g_i^{(s)} \frac{\partial^i u}{\partial x^i}\Big|_{x=0} = 0, \quad s = 1, 2, \dots, q_-, \tag{4}$$

for b) and c), and

$$\sum_{i=1}^{m-1} h_i^{(s)} \frac{\partial^i u}{\partial x^i}\bigg|_{x=1} = 0, \quad s = 1, 2, \dots, q_+, \tag{5}$$

for c).

We shall deal with the conditions for solvability of such a problem and with the correctness of its formulation. Let us note that certain analogous formulations occurred in the work of M. I. Vishik and L. A. Lyusternik ([2]) as one of the links in their theory. We shall consider this question in general form, in its own right.

We shall seek the solution of the problems posed by means of the Laplace transform. Let

$$\tilde{u}(x, \lambda) = \int_0^\infty e^{-\lambda t} u(x, t)\, dt, \qquad \tilde{f}(x, \lambda) = \int_0^\infty e^{-\lambda t} f(x, t)\, dt. \tag{6}$$

Formula (6) makes sense, for example, if $|\tilde{u}| < e^{Mt}$ and $|\tilde{f}| < e^{Mt}$, where M is some constant. The function $\tilde{u}(x, \lambda)$ will be a solution of the ordinary equation

$$L_\lambda \tilde{u} \equiv \lambda^n + \sum_{k<n,\, l\le m} A_{k,l} \lambda^k \frac{d^l \tilde{u}}{dx^l} = \tilde{f}, \tag{7}$$

which may also be written as

$$a_0(\lambda) \frac{d^m \tilde{u}}{dx^m} + a_1(\lambda) \frac{d^{m-1} \tilde{u}}{dx^{m-1}} + \dots + a_m(\lambda) \tilde{u} = \tilde{f}. \tag{8}$$

The solution of (7), with the aid of Green’ s function, is expressed in the form

$$\tilde{u}(x, \lambda) = \int_{-\infty}^{+\infty} G(x, x_1, \lambda) \tilde{f}(x_1, \lambda)\, dx_1. \tag{9}$$

Let us write the differential equation for the Green’ s function. For case a) this equation will be

$$L_\lambda G = \delta(x - x_1), \tag{10}$$

where $\delta(x - x_1)$ is the generalized Dirac function. Let

$$\frac{d^r G^{()}}{dx^r}\bigg|_{x=0} = A_r(\lambda).$$

Extend the function $G^{()}$ by zero for negative x. Computing $L_\lambda G^{()}$ by the usual method, we obtain the formula

$$L_\lambda G^{()} = \sum_{k=0}^{m-1} C_k \delta^{(k)}(x) + \delta(x - x_1), \qquad \text{where} \quad C_k = \sum_{j=0}^{m-k-1} a_{jA_{m-k-j-1}}. \quad (11)$$

In exactly the same way, for the function $G^{()}$, extended by zero outside the interval $0 \le x \le 1$, we obtain:

$$L_\lambda G^{()} = \sum_{k=0}^{m-1} C_k \delta^{(k)}(x) + \sum_{k=0}^{m-1} D_k \delta^{(k)}(x-1) + \delta(x - x_1), \quad (12)$$

where C_k is expressed by formula (11), and D_k by the formula

$$D_k = -\sum_{j=0}^{m-k-1} a_{jB_{m-k-j-1}}, \qquad \text{where} \quad \left.\frac{d^r G^{()}}{dx^r}\right|_{x=1} = B_r(\lambda). \quad (13)$$

Let us apply to the determination of G, in all cases, one more Laplace transformation with respect to the variable x. We have

$$\widehat{L_\lambda G} = \Delta(\lambda, \alpha)\, \tilde{G}, \qquad \text{where} \quad \tilde{G} = \int_{-\infty}^{+\infty} e^{-\alpha x} G(x)\, dx. \quad (14)$$

Here by $\Delta(\lambda, \alpha)$ is denoted the polynomial

$$\Delta(\lambda, \alpha) = \lambda^n + \sum_{\substack{k<n \\ l<m}} A_{k,l} \lambda^k \alpha^l.$$

The expression $L_\lambda G$ is easily computed by means of formulas (10), (11), or (12); we shall have:

$$\widehat{L_\lambda G^{(a)}} = e^{-\alpha x_1}, \qquad \widehat{L_\lambda G^{()}} = e^{-\alpha x_1} + \sum_{k=0}^{m-1} C_k \alpha^k,$$

$$\widehat{L_\lambda G^{()}} = e^{-\alpha x_1} + \sum_{k=0}^{m-1} C_k \alpha^k + e^{-\alpha} \sum_{k=0}^{m-1} D_k \alpha^k. \quad (15)$$

Consequently:

$$\widehat{G^{(a)}} = \frac{e^{-\alpha x_1}}{\Delta(\lambda,\alpha)}, \qquad \widehat{G^{()}} = \frac{e^{-\alpha x_1} + \sum_{k=0}^{m-1} C_k \alpha^k}{\Delta(\lambda,\alpha)},$$

$$\widehat{G^{()}} = \frac{e^{-\alpha x_1} + \sum_{k=0}^{m-1} C_k \alpha^k + e^{-\alpha} \sum_{k=0}^{m-1} D_k \alpha^k}{\Delta(\lambda,\alpha)}. \tag{16}$$

The first formula (16) must express the Laplace transform of $G^{(a)}$ and, consequently, must represent a regular function in some strip $M_1 < \xi < M_2$, where $\alpha = \xi + i\eta$. Choosing different strips that do not contain roots of the denominator, we obtain different values for $\widetilde{G}^{(a)}$. The Mellin integral makes it possible to reconstruct $\widetilde{G}^{(a)}$ from its image, reducing it to residues at the roots of $\Delta(\lambda,\alpha)$. The investigation shows that only one of the values obtained, namely the one corresponding to the strip containing the imaginary axis, gives the desired solution of the problem, decreasing for large values of λ. Thus the Cauchy problem is solved.

The function $\widetilde{G}^{(b)}(\alpha)$ must be the image of a function $G^{(b)}(x)$ equal to zero for $x < 0$. Computing this latter function by means of the Mellin formula and again taking into account the requirement of decrease for large values of λ, we are convinced that this is possible only in the case when $\widetilde{G}^{(b)}(\alpha)$ is regular at all roots γ_s of the equation $\Delta(\lambda,\alpha)$ lying to the right of the imaginary axis α.

We decompose $\Delta(\lambda,\alpha)$ into two factors

$$\Delta(\lambda,\alpha) = \Delta_1(\lambda,\alpha)\Delta_2(\lambda,\alpha), \quad \Delta_1(\lambda,\alpha) = \prod_{s=1}^{r_-}(\alpha-\beta_s), \quad \Delta_2(\lambda,\alpha) = \prod_{s=1}^{r_+}(\alpha-\gamma_s), \tag{17}$$

$\beta_s(\lambda)$ are the roots of $\Delta(\lambda,\alpha)$ situated in the left half-plane, and $\gamma_s(\lambda)$ are the roots of this polynomial situated in the right half-plane α. We shall call the number of roots $\beta_s(\lambda)$ the number of left influences, and the number of roots $\gamma_s(\lambda)$ the number of right influences for equation (1). Let us construct the remainder upon division of $\widetilde{G}^{(b)}$ by $\Delta_2(\lambda,\alpha)$. For this it is enough to consider a contour C_γ enclosing all roots γ_s and containing no β_s, and to compute the function

$$R_\gamma = \Delta_2(\lambda,\alpha)\left(\widetilde{G}^{(b)} - \frac{1}{2\pi i}\int_{C_\gamma} \frac{\widetilde{G}^{(b)}(\alpha')\,d\alpha'}{\alpha'-\alpha}\right). \tag{18}$$

This function will be a polynomial of degree r_+ in α, which serves as the required remainder. The coefficients R_γ are linear functions of $C_0, C_1, \dots, C_{m-1}$. Equating them to zero, we obtain a system of linear equations for C_j. These equations are independent, since each of the first coefficients $C_0, C_1, \dots, C_{r_+}$ enters into one and only one of these equations with coefficient equal to unity. In order that our problem have a solution, one must have still r_- linear relations independent of the preceding ones. Such relations are obtained from (4) in the form

$$\sum_{i=1}^{m-1} g_i^{(s)} A_i = 0, \qquad s = 1, 2, \dots, q_-. \tag{19}$$

The quantities C_j found from the resulting system will decrease exponentially for large λ. Examples show that it is possible that equations (19), in different cases, will turn out to be either dependent or independent of the rest. We obtain the main theorem.

Theorem. The mixed problem in domain b), generally speaking, is solvable and has a unique regular solution if the number of conditions q_- is equal to the number of left influences r_-. It is solvable, in particular, if the conditions have the form:

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} = 0, \qquad k = 0, 1, \dots, r_-. \tag{20}$$

Another particular case of solvability is the case when Lu is an elementary Petrovskii operator.

We pass to case c). It is easy to see that the function $\widetilde{G}^{(c)}(\alpha)$ can serve as a prototype for $G^{(c)}(x)$, equal to zero outside $(0, 1)$, only in the case when it is regular in the whole α-plane. The formula

$$R = \Delta(\lambda, \alpha) \left(\widetilde{G}^{(c)}(\alpha) - \frac{1}{2\pi i} \int_C \frac{G^{(c)}(\alpha')}{\alpha' - \alpha} \, d\alpha' \right) \tag{21}$$

gives the remainder upon division of $\widetilde{G}^{(c)}(\alpha)$ by $\Delta(\lambda, \alpha)$, which is a polynomial of degree $m-1$. All coefficients of this polynomial must be equal to zero; hence we obtain m relations for determining the constants C_k and D_k. Equations (4) and (5) give another m relations. If these relations are independent, then all the required constants can be found. It remains to consider the conditions under which the quantities C_k and D_k thus found will decrease in the right half-plane for large values of λ.

Let C be the vector with components $C_0, C_1, \dots, C_{m-1}$, and D the vector with components $D_0, D_1, \dots, D_{m-1}$. The equations obtained by setting R equal to zero can be represented in another form, by reducing them to equations expressing the vanishing at infinity of $\widetilde{G}^{(c)}$ at each of the roots of $\Delta(\lambda, \alpha)$ separately. Writing separately the terms corresponding to β_s and γ_s, we obtain the system

$$\sum_{i=1}^{m} c_i^{(s)} C_i + \sum_{i=1}^{m} d_i^{(s)} e^{-\beta_s} D_i = F_s e^{-\beta_s x_3}, \qquad s = 1, 2, \dots, r_+; \tag{22}$$

$$\sum c_i^{(s)} C_i + \sum_{i=1}^{m} d_i^{(s)} e^{-\gamma_s} D_i = F_s e^{-\gamma_s x_1}, \qquad s = r_+ + 1, r_+ + 2, \dots, m. \tag{23}$$

Multiplying each of equations (23) by $e^{-\gamma_s x_1}$ and adjoining to (22) and (23) the equations obtained from (4) and (5), we put this system in the form

$$X_1 C + Y_1 D = F_1, \qquad Y_2 C + X_2 D = F_2, \tag{24}$$

where the matrix X_1 consists of m columns and $r_+ + q_-$ rows, the matrix X_2 of m columns and $r_- + q_+$ rows; moreover, the elements of X_1 and X_2, as λ increases, remain finite. The matrices Y_1 and Y_2, as well as the matrices F_1 and F_2, decrease exponentially for large λ such that $\sigma > \sigma_0$. It is easy to verify that, if $q_- = r_-$, $q_+ = r_+$, and X_1^{-1}, X_2^{-1} do not tend to zero, then system (24) has a solution that converges for large λ and satisfies all the imposed conditions, representable in the form

$$\begin{aligned} C &= X_1^{-1} F_1 - X_1^{-1} Y_1 X_2^{-1} F_2 + X_1^{-1} Y_1 X_2^{-1} Y_2 X_1^{-1} F_1 - \cdots, \\ D &= X_2^{-1} F_2 - X_2^{-1} Y_2 X_1^{-1} F_1 + X_2^{-1} Y_2 X_1^{-1} Y_1 X_2^{-1} F_2 + \cdots. \end{aligned}$$

If $q_- \neq r_-$ and, consequently, $q_+ \neq r_+$, then the solution of system (24) gives, generally speaking, C and D growing exponentially together with λ. We obtain the theorem:

Theorem. *Problem c), generally speaking, is solvable if the number of conditions at the left end q_- is equal to the number r_- of left influences, and the number of conditions q_+ at the right end is equal to the number r_+ of right influences.*

In particular, as in the case of problem b), the problem is always solvable for the case of the simplest conditions of the form (20), and also for elementary Petrovskii operators.

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References

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2. M. I. Vishik, L. A. Lyusternik, *Uspekhi Mat. Nauk* **12**, issue 5 (77), 3 (1957).