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# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

L. G. MIKHAILOV

## INVESTIGATION OF ONE NEW TYPE OF TWO-DIMENSIONAL INTEGRAL EQUATIONS

*(Presented by Academician M. A. Lavrent'ev, 16 X 1957)*

Consider the equation\*

$$f(z) + \lambda \frac{1}{\pi z} \iint_D \frac{K(\zeta)}{\zeta - z} f(\zeta) ds = g(z), \quad (1)$$

where  $z = x + iy$ ;  $\zeta = \xi + i\eta$ ;  $ds = d\xi d\eta$ ;  $K(z)$  and  $g(z)$  are given, and  $f(z)$  is the sought, complex-valued function of real variables;  $\lambda$  is a complex parameter;  $D$  is a bounded domain whose boundary consists of a finite number of simple closed nonintersecting rectifiable lines; the origin lies inside the domain  $D$ .

Let  $\gamma_\varepsilon$  be the circle  $|z| \leq \varepsilon$  and  $D_\varepsilon = D - \gamma_\varepsilon$ . The integral in (1), and everywhere below, will be understood in the sense of the principal value, i.e., as

$$\lim_{\varepsilon \rightarrow 0} \iint_{D_\varepsilon},$$

if

$$\iint_{D_\varepsilon}$$

exists in the Lebesgue sense for every  $\varepsilon > 0$ . The class of functions integrable in the indicated sense will be denoted by  $L^*(D)$ , while retaining the notation  $L_p(D)$  for functions summable with power  $p \geq 1$ .

**1. Homogeneous equation.** Define reciprocal classes of kernels and solutions by the conditions

- A.  $K(z)f(z) \in L_p(D_\varepsilon)$ ,  $p > 2$  for every  $\varepsilon > 0$ ;
  - B.  $K(z)f(z) \in L^*(D)$ .
- (2)

Differentiating the equation with respect to  $\bar{z}$  in the sense of S. L. Sobolev <sup>(2)</sup>, and then integrating the resulting differential equation <sup>(1,3,7)</sup>, we shall have

$$f(z) = \frac{\Phi(z)}{z^2} e^{\lambda\omega(z)}, \quad (3)$$

where  $\Phi(z)$  is some holomorphic function and

$$\omega(z) = -\frac{1}{\pi} \iint_D \frac{K(\zeta)}{\zeta(\zeta - z)} ds. \quad (4)$$

If  $K(z)/z$  is not integrable, then an analytic regularizer can be introduced <sup>(3)</sup>.

Thanks to formula (3), the solution of the integral equation is reduced to finding the holomorphic function  $\Phi(z)$ . Under conditions (2), from the very

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\* The problem was posed by I. N. Vekua; in the general formulation, under the integral sign, instead of  $f(\zeta)$  there stands  $\overline{f(\zeta)}$ .

it follows from the equation <sup>(1,2,7)</sup> that  $\Phi(z)$  is continuous and holomorphic in the whole plane, except for the point  $z = 0$ , and has the expansion

$$\Phi(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \quad (5)$$

To determine  $\Phi(z)$ , we substitute (3) into the integral equation and bring it to the form

$$\frac{\Phi(z)}{z} e^{\lambda\omega(z)} = -\frac{1}{\pi} \iint_D \frac{\partial}{\partial \bar{\zeta}} \left[ \frac{\Phi(\zeta)}{\zeta} e^{\lambda\omega(\zeta)} \right] \frac{ds}{\zeta - z}. \quad (6)$$

Here we have the condition that  $\frac{\Phi(z)}{z} e^{\lambda\omega(z)}$  coincides with the integral of its derivative. It can be shown that  $\Phi(z)$  satisfies condition (6) if and only if the equalities

$$\lim_{\varepsilon \rightarrow 0} \int_{l_\varepsilon} \Phi(t) e^{\lambda\omega(t)} t^k dt = 0, \quad k = -1, 0, 1, 2, \dots, \quad (7)$$

hold, where  $l_\varepsilon$  is the circle  $|z| = \varepsilon$ . Thus, the following theorem has been obtained:

**Theorem 1.** *In order that the homogeneous integral equation (1) have a solution, it is necessary and sufficient that there exist an entire analytic function (5) satisfying conditions (7).*

**2.** Passing to the **nonhomogeneous equation**, set  $K(z)g(z) \in L_p(D)$ ,  $p > 2$ . By the substitution of the unknown function  $f = f_1 + g$  we obtain a new free term

$$g_1(z) = -\frac{1}{\pi} \iint \frac{K(\zeta)g(\zeta)}{\zeta - z} ds,$$

which is continuous and differentiable in the sense of S. L. Sobolev, as we shall assume.

Differentiating (1) with respect to  $\bar{z}$  and then integrating the resulting nonhomogeneous differential equation as in (7), we shall have

$$f(z) = g(z) + \lambda \frac{e^{\lambda\omega(z)}}{z} [\Omega(z) + \Phi(z)], \quad (8)$$

where  $\omega(z)$  and  $\Phi(z)$  are given by formulas (4) and (5), and

$$\Omega(z) = -\frac{1}{\pi} \iint \frac{K(\zeta)e^{-\lambda\omega(\zeta)}}{\zeta - z} g(\zeta) ds.$$

It may happen that  $Ke^{-\lambda\omega}g$  is not integrable, and then it is necessary to introduce an analytic regularizer.

With the help of (8), equation (1) is transformed into the equality

$$[\Omega(z) + \Phi(z)]e^{\lambda\omega(z)} = -\frac{1}{\pi} \iint_D \frac{\partial}{\partial \bar{\zeta}} [\Omega(\zeta) + \Phi(\zeta)]e^{\lambda\omega(\zeta)} \frac{ds}{\zeta - z},$$

which, like (6), expresses that the function  $(\Omega + \Phi)e^{\lambda\omega}$  coincides with the integral of its derivative, and this reduces to the conditions

$$\lim_{\varepsilon \rightarrow 0} \int_{l_\varepsilon} [\Omega(t) + \Phi(t)]e^{\lambda\omega(t)} t^k dt = 0, \quad k = 0, 1, 2, \dots \quad (9)$$

**Theorem 2.** *In order that the nonhomogeneous integral equation (1) have a solution, it is necessary and sufficient that there exist an entire analytic function (5) satisfying conditions (9).*

**3.** On the basis of Theorems 1 and 2 one can investigate various concrete cases. In Theorem 3 the discussion will concern the simplest solutions of the homogeneous equation

$$z^{-m} e^{\lambda\omega(z)}, \quad m = 2, 3, \dots, \quad (10)$$

which are obtained from (3), if for  $\Phi(z)$  one takes successively  $1, z^{-1}, z^{-2}, \dots$

**Theorem 3.** *If  $K(z)$  satisfies the Hölder condition at the point  $z = 0$ , then for every  $\lambda$  the homogeneous equation has no solutions of the form (10), while the nonhomogeneous equation has, moreover, a unique solution.*

**Proof.** We shall verify conditions (7) for the functions (10).

For all  $m = 2, 3, \dots$ , the condition

$$\lim_{\varepsilon \rightarrow 0} \int_{l_\varepsilon} e^{\lambda\omega(t)} t^{-1} dt = 0$$

is fulfilled, or

$$\lim_{r \rightarrow 0} \int_0^{2\pi} e^{\lambda\omega(r, \varphi)} d\varphi = 0. \quad (11)$$

If  $K(0) = 0$ , then  $\omega(z)$  is continuous. Passing to the limit under the integral sign in (11) gives

$$\lim_{r \rightarrow 0} \int_0^{2\pi} e^{\lambda\omega(r, \varphi)} d\varphi = e^{\lambda\omega(0)} \neq 0.$$

This shows that no function (10) can serve as a solution.

For  $K(0) \neq 0$ , the integral (4) can be represented in the form

$$\omega(z) = a(z) + K(0) \frac{\bar{z}}{z},$$

where  $a(z)$  is a continuous function. Substituting this expression for  $\omega(z)$  in (11), we again see that the condition is not fulfilled. The first part of the theorem is proved.

The solution of the nonhomogeneous equation is given by formula (8) for  $\Phi(z) \equiv 0$ . It is easy to verify that conditions (9) are fulfilled, taking into account that  $\omega(z)$  is bounded and  $\Omega(z)$  is continuous. The theorem is proved.

**Theorem 4.** *Let  $\omega(z)$  depend only on  $r = |z|$ . Then, if*

$$\lim_{r \rightarrow 0} e^{\lambda\omega(r)} \neq 0$$

*or does not exist, the homogeneous equation has no solutions different from the trivial one; but if*

$$\lim_{r \rightarrow 0} e^{\lambda\omega(r)} = 0,$$

*then it has an infinite number of linearly independent solutions (10).*

**Example.** Let  $D$  be the disk  $|z| \leq R < 1$  and

$$K(z) = \frac{1}{2 \ln r} \frac{z}{\bar{z}}.$$

Here  $K(z)$  is continuous (but does not satisfy the Hölder condition) and

$$\omega(z) = \ln \ln \frac{1}{|z|} - \ln \ln \frac{1}{R}.$$

The nonhomogeneous equation is solvable for every  $\lambda$ ; the homogeneous equation has an infinite number of solutions in the half-plane  $\operatorname{Re} \lambda > 0$  and has no solutions for  $\operatorname{Re} \lambda \leq 0$ . This shows that the principle of contraction mappings is not applicable to the operator (1).

If, following (6), one calls the index  $x$  the difference between the numbers of solutions of the given and of the transposed equations, then in our example  $x$  takes the values  $-\infty, 0, +\infty$ . The equation is neither Fredholm nor Noetherian.

**Theorem 5.** *If  $K(z)$  is continuous at the point  $z = 0$ , then the nonhomogeneous equation is solvable for every  $\lambda$ .*

**Theorem 6.** *If  $K(z)$  is continuous at the point  $z = 0$ , then the nonhomogeneous equation is solvable for*

$$|\lambda| < \frac{1}{2M},$$

where

$$M = \lim_{z \rightarrow 0} |K(z)|.$$

Let us estimate the behavior of  $\omega(z)$  at the point  $z = 0$ . If  $K(z)$  is continuous at the point  $z = 0$ , then

$$|\omega(z)| < \varepsilon \ln \frac{1}{|z|} + N(\varepsilon),$$

where  $\varepsilon > 0$  is an arbitrarily small quantity and  $N(\varepsilon)$  is a constant depending on it. If  $K(z)$  is bounded at the point  $z = 0$  and

$$M = \lim_{z \rightarrow 0} |K(z)|,$$

then

$$|\omega(z)| < 2(M + \varepsilon) \ln \frac{1}{|z|} + N_\varepsilon.$$

Substituting these estimates into (9) for  $\Phi(z) \equiv 0$ , we obtain the proofs of Theorems 5 and 6.

It is appropriate to make a comparison with one-dimensional equations. It is known that if  $K(x, t)$  satisfies the Hölder condition, then the kernel

$$\frac{K(x, t)}{x - t}$$

is singular (not Fredholm) (4), while the kernel

$$\frac{K(x, t)}{x - c}$$

can be assigned to the Fredholm—

...with (5). Of interest is the question whether the kernel

$$\frac{K(x, t)}{\sqrt{(x-t)(x-c)}}$$

is Fredholm or singular. In the two-dimensional case the kernel

$$\frac{K(\zeta, z)}{(\zeta - z)^2}$$

is likewise singular, whereas

$$\frac{K(\zeta, z)}{(z - c)^2}$$

is Fredholm. The “intermediate” case

$$\frac{K(\zeta)}{z(\zeta - z)}$$

was studied above. Theorem 3 shows that, under a Hölder condition on  $K(z)$ , this kernel is Fredholm. The example shows that already for continuous  $K(z)$  it becomes singular.

In conclusion I express my gratitude to Corresponding Member of the Academy of Sciences of the USSR I. N. Vekua for posing the problem and for valuable suggestions.

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*Note: Figure translations are in progress. See original paper for figures.*

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