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MATHEMATICS

1958

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Abstract

Full Text

MATHEMATICS

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A HYPERSURFACE IN A SPACE OF PROJECTIVE CONNECTION

(Presented by Academician P. S. Aleksandrov, 25 II 1958)

1. In the present work a differential geometry is constructed for a hypersurface in a multidimensional space of projective connection with curvature and torsion ⁽¹⁾. Principal attention is given to those objects which generalize the basic concepts of the projective-differential geometry of an ordinary surface ^(2, 3). The constructions are carried out by the group-theoretic method ⁽³⁾ and have an invariant character. Owing to the latter circumstance they also apply to hypersurfaces in spaces of Riemannian, Weyl, and affine connection.
2. A space of projective connection of N dimensions is defined ⁽¹⁾ by a system of $(N + 1)^2$ Pfaffian forms ω_K^J , subject to the structural equations

$$D\omega_K^J = [\omega_K^L \omega_L^J] + R_{K\hat{P}\hat{Q}}^J [\omega_0^{\hat{P}} \omega_0^{\hat{Q}}], \quad (1)$$

$$(J, K, L = 0, 1, \dots, N, \quad \hat{P}, \hat{Q} = 1, \dots, N).$$

The independent first integrals u^1, \dots, u^N of the completely integrable system $\omega^{\hat{P}} \equiv \omega_0^{\hat{P}} = 0$, whose rank is assumed equal to N , are local coordinates of a point $M(u)$ of the base space B . With the current point $M(u)$ of B there is associated a local projective space $P_N(u)$ of N dimensions, in which a point $M_0(u)$, conventionally identified with $M(u)$, is fixed and which is referred to a frame of $N + 1$ analytic points $M_0(u), M_1(u), \dots, M_N(u)$. The forms ω_K^J determine the principal part of the mapping of the neighboring local space $P_N(u + du)$ onto the original space $P_N(u)$ by means of the mapping of frames:

$$M_K(u + du) \rightarrow M_K(u, du) = M_K(u) + \omega_K^J M_J(u) + \rho \varepsilon_K^J M_J(u), \quad (2)$$

where $\rho = |\omega^1| + \dots + |\omega^N|$, $\lim_{\rho \rightarrow 0} \varepsilon_K^J = 0$.

The structural equations (1) ensure the invariance of the principal part of this mapping (2) with respect to transformations of the family of frames.

Remark. A space of affine connection, referred to a moving vector frame $M(u), \mathbf{e}_1(u), \dots, \mathbf{e}_N(u)$, may be regarded as a space of projective connection for which the connection forms ω_K^J are subject to the additional condition $\omega_K^0 \equiv 0$. In this case the points $M_1(u), \dots, M_N(u)$ of the projective frame determine the improper hyperplane in $P_N(u)$ and are identified with the vectors $\mathbf{e}_1(u), \dots, \mathbf{e}_N(u)$. Under the mapping (2), the improper hyperplane of the neighboring local space $P_N(u + du)$ is mapped to the improper hyperplane of the original space $P_N(u)$.

3. The tangent plane to a surface at its given point $M(u)$ is defined as the plane of the local space $P_N(u)$ onto which, in the principal part, the surface is mapped under the mapping (2).

If we assume that the origin $M_0(u) \equiv M(u)$ of the moving frame is placed on the hypersurface, and the vertices $M_1(u), \dots, M_{N-1}(u)$ are located in its tangent hyperplane, then the differential equation of the hypersurface is written as follows:

$$\omega^N = 0. \tag{3}$$

Successive prolongations of this equation give

$$\omega_i^N = \Lambda_{ij} \omega^j,$$

$$d\Lambda_{ij} - \{\Lambda_{ik} \omega_j^k - \Lambda_{kj} \omega_i^k + \Lambda_{ij} (\omega_N^N + \omega_0^0)\} = \Lambda_{ijk} \omega^k,$$

$$\begin{aligned} d\Lambda_{ijk} - \Lambda_{ij} \omega_k^l - \Lambda_{ilk} \omega_j^l - \Lambda_{ljk} \omega_i^l + \Lambda_{ijk} (\omega_N^N + 2\omega_0^0) \\ = (\Lambda_{ij} \Lambda_{lk} + \Lambda_{il} \Lambda_{jk} + \Lambda_{lj} \Lambda_{ik}) \omega_N^l - (\Lambda_{ij} \omega_k^0 + \Lambda_{ik} \omega_j^0 + \Lambda_{jk} \omega_i^0) + \Lambda_{ijkl} \omega^l, \end{aligned} \tag{4}$$

.....

$$(i, j, k, l, m, p, q, r, s = 1, \dots, N - 1).$$

A sequence of fields of fundamental objects with relative components $\Lambda_{ij}, \Lambda_{ijk}, \dots$ arises, and it is this sequence that underlies the differential geometry of the hypersurface.

The field of the fundamental object of fourth order $\Lambda_{ij}, \Lambda_{ijk}, \Lambda_{ijkl}$ is principal: it encompasses fields with arbitrary generating elements. The field following it, the fundamental object of fifth order, determines the hypersurface in a holonomic frame up to ∞^0 constants.

4. All fields of differential-geometric objects intrinsically associated with the hypersurface under study are encompassed by the fundamental fields. The construction of the encompassed fields is carried out by algebraically eliminating from the fundamental systems (4) one or another connection form according to the type of the object being encompassed. Below we give only the final formulas, by means of which the most interesting differential-geometric objects are determined successively.

A. First of all, the following relative tensors of second order are distinguished:

$$a_{ij} = \frac{1}{2}(\Lambda_{ij} + \Lambda_{ji}), \quad \bar{a}_{ij} = \frac{1}{2}(\Lambda_{ij} - \Lambda_{ji}) = R_{0ji}^N; \quad a = \text{Det } \|a_{ij}\|.$$

In the general case, when $a \neq 0$, there is also determined a symmetric tensor a^{ij} such that

$$a^{ij}a_{jk} = \delta_k^i.$$

B. Next, systems of quantities of third order are constructed successively:

$$M_{ijk} = \frac{1}{2}(\Lambda_{ijk} + \Lambda_{jik}), \quad \Lambda_{ijk} = \frac{1}{3}(M_{ikj} + M_{ikj} + M_{kji}),$$

$$b_k = a^{ij}\tilde{\Lambda}_{ijk}, \quad \tilde{b}_k = a^{ij}M_{kij}, \quad \hat{b}_k = a^{ij}M_{ijk} - \tilde{b}_k,$$

$$b_{ijk} = (N+1)\tilde{\Lambda}_{ijk} - a_{ij}b_k - a_{ik}b_j - a_{jk}b_i, \quad b_{ij} = a^{pq}a^{rs}b_{ipr}b_{jq_s},$$

$$b_0 = a^{ij}b_{ij}, \quad b = \det \|b_{ij}\|.$$

Generally speaking, each of these systems of quantities only together with the components Λ_{ij} of the fundamental object of second order determines an object of third order. However, the systems of quantities b_{ijk} and b_{ij} themselves determine two relative tensors, while the quantities b_0 and b are relative invariants.

C. The objects of fourth order that interest us are encompassed by objects arising from prolongations of the previously constructed objects of third order.

First, the differential prolongation of the invariant $\ln b_0$ (in the general case $b_0 \neq 0$) leads to the quantities c_k :

$$d \ln b_0 = \omega_N^N - \omega_0^0 + c_k \omega^k.$$

With their aid the following linear objects are constructed:

$$h_k = \frac{1}{2}c_k + \frac{1}{2(N+1)}(\tilde{b}_k + \tilde{\tilde{b}}_k), \quad j^i = a^{ik} \left(\frac{1}{N+1}\tilde{b}_k - \eta h_k \right),$$

$$\tilde{b}_{pqk} = (N+1)M_{pqk} - a_{pk}\tilde{b}_q - a_{qk}\tilde{b}_p - a_{pq}(\tilde{b}_k + \tilde{\tilde{b}}_k) - (N+1)(a_{pk|q} + a_{qk|p}),$$

$$\tilde{b}_{ij} = a^{pq}a^{rs}\tilde{b}_{ipr}\tilde{b}_{jqs}.$$

Moreover, in the general case, when the tensor \tilde{b}_{ij} is nondegenerate, a tensor \tilde{b}^{jk} is determined such that $\tilde{b}_{ij}\tilde{b}^{jk} = \delta_i^k$.

Secondly, prolongation of the quantities \tilde{b}_i leads to the quantities l_{ij} :

$$d\tilde{b}_i = \tilde{b}_k\omega_i^k - b_i\omega_0^0 + (N+1)(a_{ik}\omega_N^k - \omega_i^0) + l_{ij}\omega^j.$$

With their aid the following linear objects are constructed (a_{ij}, \tilde{b}_i, l) , l^i , k_j , \hat{h}_j , \hat{j}^i :

$$l = \frac{1}{N-1}a^{ij} \left(l_{ij} - \frac{1}{N+1}\tilde{b}_i\tilde{b}_j \right),$$

$$l^i = \tilde{b}^{il}a_{pr}a_{qs}\tilde{b}_{ipq} \left\{ l_{rs} - \frac{1}{N+1}\tilde{b}_r\tilde{b}_s - \frac{1}{2}(l\Lambda_{rs} + la_{rs}) - a_{rk}l^k\tilde{b}_s - \frac{N+1}{2}a_{rs}a_{kl}j^k j^l \right\},$$

$$k_j = \frac{1}{N+1}\tilde{b}_j - a_{ji}l^i, \quad \hat{h}_j = h_j - k_j, \quad \hat{j}^i = j^i - l^i.$$

Moreover, linear objects $\{n^i(\tau), n(\tau)\}$, $\{m_j(\tau), m(\tau)\}$, depending on the invariant parameter τ , are constructed:

$$n^i(\tau) = l^i + \tau\hat{j}^i, \quad n(\tau) = \frac{1}{2(N+1)}l - \frac{1}{N+1}n^k\tilde{b}_k + \frac{1}{2}a_{ij}n^i n^j,$$

$$m_j(\tau) = k_j + \tau\hat{h}_j, \quad m(\tau) = n(\tau) + n^k(\tau)m_k(\tau).$$

. Prolongation of the object $m_i(\tau)$ leads to the quantities $p_{ij}(\tau)$:

$$dm_i(\tau) = m_k(\tau)\omega_i^k - m_i(\tau)\omega_0^0 - \omega_i^0 + p_{ij}(\tau)\omega^j.$$

With their aid a tensor $\partial_{ij}(\tau)$, depending on the parameter τ , is constructed:

$$\partial_{ij}(\tau) = p_{ij}(\tau) - m_i(\tau)m_j(\tau) + \Lambda_{ij}m(\tau).$$

5. We shall now give a geometric interpretation of the differential-geometric objects constructed, indicating those geometric images which are determined by them in the local spaces. For these images we shall retain the names that they acquire when the enveloping space becomes three-dimensional projective space.

. First of all, two relatively invariant forms are determined: $a_{ij}\omega^i\omega^j$ and $b_{ijk}\omega^i\omega^j\omega^k$. Their vanishing singles out in the tangent hyperplane, respectively, the cone of asymptotic directions and the cone of Darboux directions.

. The ratio of the forms $b_{ijk}\omega^i\omega^j\omega^k/a_{ij}\omega^i\omega^j$ is absolutely invariant and is the generalized linear element of Fubini.

. The forms $\hat{h}_k\omega^k$, $\partial_{ij}(\tau)\omega^i\omega^j$ are absolutely invariant. The equations $\hat{c}_k\omega^k = 0$ and $\partial_{ij}(0)\omega^i\omega^j = 0$ determine in the tangent hyperplane, respectively, the canonical tangent plane and the cone of Fubini directions.

. In the local space $P_N(u)$, associated with the current point $M(u)$ of the hypersurface, a pencil of invariant Darboux hyperquadrics is determined:

$$a_{ij}x^i x^j + \frac{2}{N+1}\tilde{b}_i x^i x^N - 2x^0 x^N + \left(\frac{1}{N+1}l + \delta b_0\right)x^N x^N = 0.$$

In this equation of the pencil, x^0, \dots, x^N are the coordinates of the current point relative-

respectively, the frame $M_0(u), \dots, M_N(u)$, and σ is an invariant parameter of the pencil. For $\sigma = 0$, Li' s hyperquadric is distinguished from the pencil.

D. With the current point $M(u)$ of the hypersurface there is associated a one-parameter family of invariant points

$$Q(\tau) = n(\tau)M_0 - n^k(\tau)M_k + M_N,$$

where τ is an invariant parameter. This family, together with the initial point $M_0(u)$, determines the canonical pencil of projective normals. For $\tau = 0$, the Wilczynski directrix is singled out from the pencil, and for $\tau = 1$, the Fubini projective normal.

E. In the local space $P_N(u)$ there is also defined a one-parameter family of invariant hyperplanes

$$x^0 - m_k(\tau)x^k - m(\tau)x^N = 0.$$

Together with the tangent hyperplane they determine a one-parameter family of second normals.

6. A hypersurface in an N -dimensional space of projective connection may be regarded as the base of an $(N - 1)$ -dimensional space of projective connection, whose local spaces are the tangent hyperplanes, while the connection-defining mapping is obtained by carrying out mapping (2) and projecting the image of the neighboring tangent hyperplane onto the original one from some fixed point of the original local space. Each such connection is determined by the forms ω^i , $\tilde{\omega}_j^i = \omega_j^i + \Pi_{jk}^i \omega^k$, $\tilde{\omega}_0^0 = \omega_0^0 + \Pi_{0k}^0 \omega^k$, $\tilde{\omega}_j^0 = \omega_j^0 + \Pi_{jk}^0 \omega^k$, in which the coefficients Π_{jk}^i , Π_{0k}^0 , Π_{jk}^0 are components of an object of projective connection. A two-parameter family of such connections, invariantly attached to the hypersurface, is determined by the formulas

$$\Pi_{jk}^i = \Lambda_{jk} n^i(\tau), \quad \Pi_{0k}^0 = \rho \hat{h}_k, \quad \Pi_{jk}^0 = -\Lambda_{jk} n(\tau) - \rho m_j(\tau) \hat{h}_k.$$

A hypersurface in a space of projective connection also carries an infinite set of affine connections, determined by the forms ω^i , $\hat{\omega}_j^i = \omega_j^i + \Gamma_{jk}^i \omega^k - \delta_j^i \omega_0^0$, in which the coefficients Γ_{jk}^i are components of an object of affine connection. Setting

$$\Gamma_{jk}^i = \Lambda_{jk} n^i(\tau) + \delta_j^i m_k(\rho) + \delta_k^i m_j(\sigma),$$

we obtain a three-parameter family of affine connections intrinsically attached to the hypersurface.

Remark. All the objects obtained and the formulas defining them are preserved also for a hypersurface of a space of affine connection; in this case only their differential equations are simplified because of the condition $\omega_j^0 \equiv 0$. In particular, owing to this condition, in a space of affine connection the system of quantities $\tilde{b}_i a^{ik}$ is a system of components of the linear object determining the vector of the generalized affine Blaschke normal:

$$\mathbf{n} = \frac{\tilde{b}_i a^{ik}}{N + 1} \mathbf{e}_k(u) + \mathbf{e}_N(u).$$

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Received
11 II 1958

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