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MATHEMATICS

1958

SovietRxiv

Abstract

Full Text

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EQUATIONS OF OSCILLATIONS WITH SLOWLY VARYING PARAMETERS

(Presented by Academician N. N. Bogolyubov, 21 II 1958)

MATHEMATICS

§ 1. **Statement of the problem.** Consider the system

$$\frac{d}{dt}\left[m(\vec{\mu})\dot{x}\right] + Q(\vec{\mu}, x) = \varepsilon f(x, \dot{x}, \vec{\mu}), \qquad \dot{\vec{\mu}} = \varepsilon \vec{\varphi}(x, \dot{x}, \vec{\mu}), \tag{1}$$

where ε is a small parameter; $\vec{\mu} = \{\mu_1, \mu_2, \dots, \mu_n\}$; $\vec{\varphi} = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$. In this system the first equation is regarded as the equation of oscillations with one degree of freedom of a variable mass $m > 0$, with coordinate x and velocity $\dot{x}$, under the action of a force Q and a small perturbation εf; $\vec{\mu}$ represents the set of n parameters $\mu_1, \mu_2, \dots, \mu_n$, on which m, Q, f depend. The equation $\dot{\vec{\mu}} = \varepsilon \vec{\varphi}$ describes the variation of the parameters μ_i $(i = 1, 2, \dots, n)$ of the oscillatory system; ε characterizes the smallness of the perturbing force εf and the slowness of the variation of μ_i, since $\dot{\mu}_i \sim \varepsilon$. If $\vec{\varphi}$ does not depend on $x, \dot{x}$, then the equation $\dot{\vec{\mu}} = \varepsilon \vec{\varphi}$ is integrated independently of the equation of oscillations, and its solution is $\vec{\mu} = \vec{\mu}(\varepsilon t)$. Then the first equation (1) becomes an equation of the type

$$\frac{d}{dt}\left[m(\tau)\dot{x}\right] + Q(\tau, x) = \varepsilon f(x, \dot{x}, \tau) \quad (\tau = \varepsilon t - \text{“slow time”}\ ),$$

which was studied in (^{3},^{5} ,^{8}). In the general case $\partial\vec{\varphi}/\partial x,\ \partial\vec{\varphi}/\partial\dot{x} \neq 0$, and then (1) is a generalization of the problem mentioned above.

Consider a solution of (1) satisfying the initial conditions $x(0) = x_0,\ \dot{x}(0) = \dot{x}_0\ (x_0^2 + \dot{x}_0^2 \neq 0),\ \vec{\mu}(0) = \vec{\mu}_0$ on a large interval $t \sim 1/\varepsilon$. Suppose that in the region under consideration $\operatorname{sign} Q = \operatorname{sign} x$ and that the equation $m(\vec{\mu})\ddot{x} + Q(\vec{\mu}, x) = 0$ (here $\vec{\mu}$ is an independent parameter) has, in the corresponding domain of variation of the initial values and of $\vec{\mu}$, only periodic solutions. Under these conditions and certain other restrictions, $x(t, \varepsilon)$—the solution of (1)—for sufficiently small ε, on the interval $t \sim 1/\varepsilon$, oscillates about the equilibrium position $x = 0$, with maxima and minima alternating. It is required to derive formulas for the amplitude of the oscillations and the parameter $\vec{\mu}$.

§ 2. **Main results.** There exist functions $F_1(\varepsilon t) < 0$, $F_2(\varepsilon t) < 0$, and $\vec{\mu}(\varepsilon t)$ possessing the following properties: on the interval $t \sim 1/\varepsilon$, F_1 and F_2, with error $\sim \varepsilon$, describe respectively the maxima and minima of $x(t, \varepsilon)$, while $\vec{\mu}(\varepsilon t)$, with the same accuracy, describes the variation of $\vec{\mu}$. The functions F_1, F_2, and $\vec{\mu}$ are connected by the finite equation

$$\int_{F_2}^{F_1} Q(\vec{\mu}, x)\, dx = 0, \tag{2}$$

and the remaining $(n+1)$ relations for determining the unknown functions F_k, μ_i $(k = 1, 2;\ i = 1, 2, \dots, n)$ are defined by the system of equations

$$\dot{I} = \varepsilon A + \varepsilon B, \qquad \dot{\vec{\mu}} = \varepsilon \vec{\varphi}, \tag{3}$$

where

$$I \equiv 2m^{1/2} \sum_{k=1}^{2} (-1)^{k+1} \int_0^{F_k} dx \left(2 \int_x^{F_k} Q(\vec{\mu}, z)\, dz \right)^{1/2}; \tag{4}$$

$$A \equiv \sum_{i,k=1}^{2} (-1)^{i+k+1} \int_0^{F_k} f \left[x, (-1)^i m^{-1/2} \left(2 \int_x^{F_k} Q(\vec{\mu}, z)\, dz \right)^{1/2}, \vec{\mu} \right] dx; \tag{5}$$

$$B \equiv m^{-1/2} \sum_{i,k=1}^{2} (-1)^{k+1} \int_0^{F_k} \left\{ \vec{\varphi} \left[x, (-1)^i m^{-1/2} \left(2 \int_x^{F_k} Q(\vec{\mu}, z)\, dz \right)^{1/2}, \vec{\mu} \right] - \bar{\vec{\varphi}} \right\}$$
$$\times \left(2 \int_x^{F_k} Q(\vec{\mu}, z)\, dz \right)^{-1/2} \int_0^x \nabla_\mu [Q(\vec{\mu}, z) m(\vec{\mu})]\, dz; \tag{6}$$

$$\bar{\vec{\varphi}} \equiv T^{-1} m^{1/2} \sum_{i,k=1}^{2} (-1)^{k+1} \int_0^{F_k} \vec{\varphi} \left[x, (-1)^i m^{-1/2} \left(2 \int_x^{F_k} Q(\vec{\mu}, z)\, dz \right)^{1/2}, \vec{\mu} \right]$$
$$\times \left(2 \int_x^{F_k} Q(\vec{\mu}, z)\, dz \right)^{-1/2} dx; \tag{7}$$

$$T \equiv 2m^{1/2} \sum_{k=1}^{2} (-1)^{k+1} \int_0^{F_k} dx \left(2 \int_x^{F_k} Q(\vec{\mu}, z)\, dz \right)^{-1/2}, \tag{8}$$

where $\nabla_\mu = \{\partial/\partial\mu_1, \partial/\partial\mu_2, \dots, \partial/\partial\mu_n\}$ is the Hamiltonian operator. The initial values for (3) are determined by the equations

$$2\int_{x_0}^{F_k(0)} Q(\vec{\mu}_0, x)\,dx = m(\vec{\mu}_0)\dot{x}_0^2, \qquad \vec{\mu}(0) = \vec{\mu}_0, \qquad \operatorname{sign} F_k(0) = (-1)^{k+1} \quad (k = 1, 2), \tag{9}$$

where $x_0, \dot{x}_0, \vec{\mu}_0$ are the prescribed initial values for the solution of (1). The period of oscillations T is expressed, with an error $\sim \varepsilon$, by formula (8), in which t is taken to be an arbitrary instant of time within the period under consideration.

To find F_k, μ_i ($k = 1, 2;\ i = 1, 2, \dots, n$), one should find the initial values from (9), eliminate one of the unknown functions from (3) by means of (2), and then solve (3) with the initial values thus found.

In the general case $x(t, \varepsilon)$ oscillates within asymmetric limits, i.e. $F_1 \neq -F_2$; $\dot{x}$ also oscillates, but in the approximation under consideration always within symmetric limits. The amplitude of $\dot{x}$ is equal to $\pm\sqrt{2E/m}$, where

$$E = \int_0^{F_1} Q(\vec{\mu}, x)\,dx = \int_0^{F_2} Q(\vec{\mu}, x)\,dx. \tag{10}$$

After finding F_k, μ_i, one can find E and compute the amplitude $\dot{x}$, and also find the period from formula (8). Transforming (3), one can derive equations for the direct calculation of E:

$$\dot{E} = \varepsilon\{(A + B_1)T^{-1} - m^{-1}E\bar{\vec{\varphi}}\nabla_\mu m\}, \qquad \dot{\vec{\mu}} = \varepsilon\bar{\vec{\varphi}}, \tag{11}$$

where

$$\begin{aligned} B \equiv m^{-1/2} \sum_{i,k=1}^{2} (-1)^{k+1} \int_0^{F_k} \vec{\varphi} \left[x, (-1)^i m^{-1/2} \left(2\int_x^{F_k} Q(\vec{\mu}, z)\,dz \right)^{-1/2}, \vec{\mu} \right] \\ \times \left(2\int_x^{F_k} Q(\vec{\mu}, z)\,dz \right)^{-1/2} \int_0^x \nabla_\mu [Q(\vec{\mu}, z) m(\vec{\mu})]\,dz, \end{aligned}$$

and A, $\vec{\varphi}$, T are determined by formulas (5), (7), (8). Equations (11), independently of (3), can be regarded as equations for F_k, μ_i; however, in the form (3) these equations are simpler and in most cases are more easily integrated. The developed methods make it possible to find F_k, μ_i with high accuracy.

§ 3. **Physical interpretation of the results; examples.** The integral $I = \oint p\,dp = \int_T m\dot{x}^2 dt$ $(p = m\dot{x},\ \ q = x)$ is called the action integral. If in the first equation (1) one puts $\vec{\mu} = \text{const}$, $f \equiv 0$, then one obtains the generating equation $m(\vec{\mu})\ddot{x} + Q(\vec{\mu}, x) = 0$ with Hamiltonian function

$$H = \frac{p^2}{2m(\vec{\mu})} + \int_0^q Q(\vec{\mu}, z)\,dz.$$

For solutions of the generating equation $I = \text{const}$, while for (1) I is, generally speaking, a variable quantity, since the amplitude x and the parameter $\vec{\mu}$ vary with time. (4) represents, with an error $\sim \varepsilon$, the action integral for (1); εA, according to (5), approximately represents the work of the perturbing force over a period:

$$\varepsilon A = \varepsilon \int_T f\dot{x}\,dt;$$

εB, according to (6), can be written in the form

$$\varepsilon B = \varepsilon \int_T (\vec{\varphi} - \bar{\vec{\varphi}})\nabla_\mu H\,dt.$$

Since $\dot{\vec{\mu}} = \varepsilon\vec{\varphi}$, $\varepsilon(\vec{\varphi} - \bar{\vec{\varphi}})$ is the deviation of the velocity of change of $\vec{\mu}$ from its averaged value, since, according to (7),

$$\bar{\vec{\varphi}} = T^{-1} \int_T \vec{\varphi}\,dt.$$

Consequently, εB, accurate up to ε inclusively, is the virtual change of H over a period, caused by a virtual change of $\vec{\mu}$ with velocity $\varepsilon(\vec{\varphi} - \bar{\vec{\varphi}})$. Thus, (3) is written in the form

$$\dot{I} = \varepsilon \int_T f\dot{x}\,dt + \varepsilon \int_T (\vec{\varphi} - \bar{\vec{\varphi}})\nabla_\mu H\,dt, \qquad \dot{\vec{\mu}} = \varepsilon\bar{\vec{\varphi}}. \tag{12}$$

System (12) describes the change of the action integral I. The velocity of its change is equal to the work of the perturbing force εf, added to the virtual work of the change of $\vec{\mu}$.

If $f \equiv 0$, i.e. there is no perturbation, and $\vec{\varphi} = \bar{\vec{\varphi}}$, i.e. the parameter $\vec{\mu}$ changes not only slowly but also with slowly varying velocity

$$\dot{\vec{\mu}} = \varepsilon\vec{\varphi}\,[F_k(\varepsilon t), \vec{\mu}(\varepsilon t)],$$

then (12) ((3)) has the integral $I = \text{const}$, i.e. in this case I is an adiabatic invariant and over the time $t \sim 1/\varepsilon$, with accuracy up to ε, does not change. This result is known in the physical literature (^{1}, ^{2}). In a number of cases, when $f \not\equiv 0$, $\vec{\varphi} \neq \bar{\vec{\varphi}}$, I can also be an adiabatic invariant, if

$$\varepsilon \int_T \{f\dot{x} + (\vec{\varphi} - \bar{\vec{\varphi}})\nabla_\mu H\}\, dt \equiv 0.$$

In the general case, however, as (12) ((3)) shows, I is not an invariant and over the time $t \sim 1/\varepsilon$ can change, generally speaking, by a finite amount. (10) shows that E is an approximate expression for the total energy of the oscillations (1). Equation (11) can be written in the form

$$\dot{E} = \varepsilon T^{-1} \int_T f\dot{x}\, dt + \varepsilon T^{-1} \int_T \vec{\varphi}\nabla_\mu H\, dt, \qquad \dot{\vec{\mu}} = \varepsilon\bar{\vec{\varphi}}. \tag{13}$$

Equation (13) describes the change of energy: the velocity of its change is equal to the average over a period of the power of the force εf, added to the power expended on changing $\vec{\mu}$.

If the solution of the generating equation is known in explicit form, then the asymptotics of (1) can be obtained directly from (12) or (13). In the general case the solution of
$m(\vec{\mu})\ddot{x} + Q(\vec{\mu}, x) = 0$ cannot be written as an explicitly known function of t, of arbitrary constants, and of $\vec{\mu}$; then one should use system (3) (or (11)), for whose integration knowledge of this solution is not required.

The equations describing the change of the action integral, the energy, and the parameter $\vec{\mu}$ on the interval $t \sim 1/\varepsilon$, written in the form (12), (13), are valid not only for system (1), whose generating equation has a Hamiltonian function of the special form
$H = \dfrac{p^2}{2m} + \displaystyle\int_0^q Q\, dx$, but also for more general systems of the form
$\dot{q} = \partial H/\partial p + \varepsilon F(p, q, \vec{\mu})$,
$\dot{p} = -\partial H/\partial q + \varepsilon f(p, q, \vec{\mu})$,
$\dot{\vec{\mu}} = \varepsilon\vec{\varphi}(p, q, \vec{\mu})$, where $H(p, q, \vec{\mu})$ is an arbitrary Hamiltonian function of the generating system
$\dot{q} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial q$ ($\vec{\mu} = \text{const}$), having periodic solutions. These results can also be extended to some oscillatory systems with many degrees of freedom.

Let us consider, as examples, two systems:

$$\text{a)} \quad \ddot{x} + \mu^2 x = 0, \quad \dot{\mu} = \varepsilon\dot{x}^2; \qquad \text{b)} \quad \ddot{x} + \mu^2 x = 0, \quad \dot{\mu} = \varepsilon x^2. \tag{14}$$

Here μ is the natural frequency of the oscillations, whose change depends on $\dot{x}$ or x. For (14) $F_1 = -F_2 = F$. Forming (3) and integrating, we find, respectively:

a) $F = \frac{1}{c_1}\left(\frac{\varepsilon t}{4} + c_2\right)^{-1/2}$,
$\mu = c_1^2\left(\frac{\varepsilon t}{4} + c_2\right)^2$, i.e. $F \sim \mu^{-3/4}$;
b) $F = (c_1 \varepsilon t + c_2)^{-1/6}$,
$\mu = \frac{3}{4c_1}(c_1 \varepsilon t + c_2)^{2/3}$,
$F \sim \mu^{-1/4}$. From (3) it follows that, for (14), $I \neq$ const. Now, for comparison, take the equation
$\ddot{x} + k^2 x = 0$, where:
a) $k \equiv \mu(\varepsilon t) = c_1\left(\frac{\varepsilon t}{4} + c_2\right)^2$;
b) $k \equiv \mu(\varepsilon t) = \frac{3}{4c_1}(c_1 \varepsilon t + c_2)^{2/3}$.
For this equation I is an adiabatic invariant and $F \sim \mu^{-1/2}$ (this is the known result of the WKB method). Thus, if in (14) one replaces the slowly varying frequency μ by its zeroth approximation, an entirely different law of amplitude variation is obtained.

The restrictions under which the results presented are valid basically coincide with the conditions stated in (8), and therefore we shall not formulate them in detail here. We note only that, in addition to these conditions, it is sufficient to require for $\vec{\varphi}$ continuity and continuous differentiability with respect to $x, \dot{x}, \mu_i$ in the domain under consideration, and for m, Q, f the existence of such derivatives with respect to μ_i as are indicated in (8) for the argument $\tau = \varepsilon t$. We note that systems similar in form to (1) were considered in ([6],[7]).

The author takes this opportunity to express his gratitude to Academician N. N. Bogolyubov for his attention to the present work.

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Received
19 II 1958

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Note: Figure translations are in progress. See original paper for figures.

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