SovietRxiv

Soviet-era science, translated into English

MATHEMATICS

N. M. KOROBOV

1958

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-195801.79935

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

N. M. KOROBOV

ON THE ESTIMATION OF RATIONAL TRIGONOMETRIC SUMS

(Presented by Academician I. M. Vinogradov, 4 VII 1957)

Let us denote by S the sum

$$S = \sum_{x=1}^{P} e^{2\pi i \frac{a_1 x + \cdots + a_{n+1} x^{n+1}}{q}}, \tag{1}$$

where $q, a_1, \ldots, a_{n+1}$ are integers and $(q, a_{n+1}) = 1$. In various problems of number theory one needs nontrivial estimates for the modulus of the sums (1), and it is important to have as sharp as possible estimates valid for values of P from as wide an interval as possible.

From I. M. Vinogradov' s theorem ([1]) on trigonometric sums of general type it follows that, for $P = q^{1/r}$, there exist positive constants c_1 and α_1 such that on the interval

$$1 < r \leq n \tag{2}$$

the estimate

$$|S| \leq e^{c_1 n \ln^2 n} P^{1 - \frac{\alpha_1}{n^2 \ln n}} \tag{3}$$

holds.

This estimate, with slight changes, extends to the intervals $0 < r \leq 1$ and $n < r < n + 1$. For $r \geq n + 1$ there obviously exist sums S for which the trivial estimate cannot be improved. Thus, the interval of values of P on which estimates of type (3) have been obtained is extremely wide, and one can raise only the question of refining these estimates.

As shown in ([2]), there exist estimates of the sums S more precise than the estimates (3), but they were found only for the interval $1 < r < 1 + \frac{1}{n}$, which constitutes a small part of the interval (2). In the present work it is possible, throughout the whole interval $1 < r \leq n$, to replace in estimate (3) the coefficient $e^{c_1 n \ln^2 n}$ by an absolute constant C and at the same time, on a significant part

SovietRxiv

of this interval, to improve the "lowering factor" $P^{\frac{\alpha_1}{n^2 \ln n}}$ to the value $P^{\frac{\alpha}{n^2}}$. An improvement of estimate (3) for n growing together with P is also obtained for the intervals $0 < r \leq 1$, $n < r < n+1$.

Let $(q, a_{n+1}) = 1$ and $1 \leq n < p_1 - 1$, where p_1 is the smallest prime divisor of q. Then the following theorems hold:

Theorem 1. There exist absolute constants C and α such that, for $P = q^{1/r}$, on the interval $1 < r < n+1$ the estimate

$$|S| \leq CP^{1-\frac{\alpha r(n+1-r)}{n^4 l^2}},$$

holds, where $l = \ln \frac{2n}{n+1-r}$.

Theorem 2. Whatever fixed $\varepsilon > 0$ may be, there exist a constant $\alpha = \alpha(\varepsilon)$ and an absolute constant C such that, for $P = q^{1/r}$, on the interval $\varepsilon n < r < n - \varepsilon n$ the estimate

$$|S| \leq CP^{1-\frac{\alpha}{n^2}}$$

holds.

Theorem 2, obviously, follows from Theorem 1. In the proof of Theorem 1 a new approach to estimates of trigonometric sums is applied, in which the rationality of the sums under consideration is used. In addition, the following theorem is used essentially.

Theorem 3. Let n, r, k, τ, q, and P be integers satisfying the conditions

$$1 \leq r \leq n; \qquad \tau \geq 1; \qquad k > n^2 + n\tau; \qquad q > 2k; \qquad \left(\frac{q}{2k}\right)^{\frac{1}{r+1}} < P \leq \left(\frac{q}{2k}\right)^{\frac{1}{r}}.$$

Then there exists an absolute constant c such that, for the number $N_k(P)$ of solutions of the system of congruences

$$\left.\begin{aligned} x_1 + \dots + x_k &\equiv y_1 + \dots + y_k, \\ x_1^2 + \dots + x_k^2 &\equiv y_1^2 + \dots + y_k^2, \\ &\dots \\ x_1^n + \dots + x_k^n &\equiv y_1^n + \dots + y_k^n \end{aligned}\right\} \pmod q \qquad (1 \leq x_\nu, y_\nu \leq P) \tag{4}$$

the estimate

$$N_k(P) \leq e^{c(n+\tau)k \ln k}\, P^{2k - \frac{r(2n+1-r)}{2} + \frac{\tau(n+1)}{2}\left(1-\frac{1}{n}\right)^\tau} \tag{5}$$

is valid.

SovietRxiv

The proof of Theorem 3 is carried out by the method of I. M. Vinogradov. It is easy to show that, for the number of solutions of the system of congruences (4), the following lower estimate is valid:

$$N_k(P) > C(k)\, P^{2k - \frac{r(2n+1-r)}{2}},$$

where $C(k)$ is some positive constant depending only on k. From a comparison of this estimate with estimate (5) it is seen that, for sufficiently large values of τ, the assertion of Theorem 3 can no longer be essentially strengthened.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
27 VI 1957

REFERENCES

1. I. M. Vinogradov, *Izv. AN SSSR, ser. matem.*, **14**, 199 (1950).

2. Hua Lo-keng, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **22**, 16 (1947).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.