# Mathematics

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**Abstract**

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# ON A METHOD FOR COUNTING INTEGER POINTS IN $n$-DIMENSIONAL POLYHEDRA

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We shall assume that in $n$-dimensional Euclidean space a rectangular Cartesian coordinate system is given, defining an $n$-dimensional cubic integral lattice. To each lattice point we assign the number one, calling it the **full weight of the integer point**. Each of the $2n$ half-lines issuing from an integer point parallel to one or another of the coordinate axes will be called a **ray** (or a **one-dimensional extension**) of this point, and to it we assign $1/2n$ of the full weight of the point itself. Each of the $2(n-1)$ pairwise opposite two-dimensional quadrants issuing from a ray parallel to one or another of the coordinate two-dimensional planes will be called a **two-dimensional extension** of the ray, and to it we assign $1/2(n-1)$ of the weight corresponding to the ray, i.e. $1/4n(n-1)$ of the full weight of the point itself. Each of the $2(n-2)$ three-dimensional octants issuing from a two-dimensional extension parallel to one or another of the three-dimensional coordinate planes will be called a **three-dimensional extension** of the two-dimensional extension, assigning to it $1/8n(n-1)(n-2)$ of the full weight of the original integer point, and so on, so that to a $k$-dimensional extension of a $(k-1)$-dimensional extension there will correspond $1/2^k n(n-1)\dots(n-k+1)$ of the full weight of the given integer point. Let us note that every $(n-2)$-dimensional extension will have four $(n-1)$-dimensional extensions, while every $(n-1)$-dimensional extension has two (directly opposite) $n$-dimensional extensions, and to each of the latter corresponds $1/2^n n!$ of the full weight of the original integer point.

Having in the space under consideration some $n$-dimensional polyhedron, we shall agree to call the **skeleton of the polyhedron** the collection of lattice points located both strictly inside the polyhedron (which, in order not to complicate the notion "inside," we shall regard as non-self-intersecting) and on its boundary, i.e. on its faces of various dimensions, including one-dimensional faces (edges) and zero-dimensional faces (vertices). To each interior point of the skeleton we assign its full weight—one; and the full weight of each boundary point we split into an "interior," with respect to the given polyhedron, weight and an "exterior" weight, and to every such point, as a point of the skeleton, we shall assign its interior weight. This will be done in the following way.

If an integer point lies at a “true” vertex $P$ of the polyhedron (in the terminology of A. D. Aleksandrov), i.e. at a point where $(n-1)$-dimensional faces meet in a number not less than $n$, then:

1) One must determine how many rays issue from $P$ strictly into the interior of the polyhedron, and assign to each of them the fraction $1/2n$, i.e. the portion of the interior weight of the point $P$ corresponding to such a ray.

2) One must determine how many rays issue from $P$ strictly into the interior of the $(n-1)$-dimensional faces of the polyhedron meeting at $P$. To each of them one must assign the fraction $1/4n$, guided by the following considerations.

Strictly inside the polyhedron, from such a ray there will emanate some number $k$ (where necessarily $1 \leq k \leq n-1$) of two-dimensional extensions of the ray, while in the face itself there remain $2(n-k-1)$ two-dimensional extensions (the remaining $k$ extensions will be strictly exterior with respect to the polyhedron). The three-dimensional extensions of the two-dimensional extensions lying in the face itself, considered together with all their subsequent extensions, are divided equally between the interior and the exterior of the polyhedron (half of them will go strictly inside, and half strictly outside the polyhedron), so that, if one adds the fractions corresponding to strict entrances into the polyhedron of the various successive extensions, beginning with the two-dimensional and ending with the $n$-dimensional, one obtains in the end

$$\frac{k}{4n(n-1)} + \frac{1}{2}\frac{2(n-k-1)}{4n(n-1)} = \frac{1}{4n},$$

and this will be the share of the inner weight of the point $P$ corresponding to each such ray.

3) For $n-2 \geq m \geq 1$ rays issuing from $P$ strictly into $m$-dimensional faces, it is no longer possible to assign a similar constant fraction for a given $m$ (unless one counts such polyhedra of a special form as rectangular parallelepipeds with edges parallel to the coordinate axes). Therefore here the share of the inner weight of the point $P$ must be counted separately for each $m$-dimensional face into which certain rays of the point $P$ go strictly. In doing so one must be guided by the following properties of extensions. First, the strict inward entrance of a ray and of the subsequent extensions generated by it into an $m$-dimensional face excludes the possibility of their strict inward entrance into faces of dimension higher than $m$, and of entrance at all into faces of dimension lower than $m$, as well as into other $m$-dimensional faces. Second, if some extension or other goes (from the extension generating it) strictly inside a certain face, then, by branching through the subsequent extensions generated by it, it reaches (if not directly, then through faces of higher dimensions that converge in the given face) such extensions as necessarily go either strictly inside

or strictly outside the polyhedron itself. In other words, there can be no extension which, having strictly entered some face, would not leave it completely through its subsequent extensions: partly strictly inside and partly strictly outside the polyhedron. Thus, in computing the share of the inner weight of an integral vertex corresponding to a separate one of its rays that has gone strictly inside some face, one must trace all exits of the ray through its subsequent extensions strictly into the polyhedron, and assign to each such exit a fraction corresponding to the dimension of the exiting extension (to the exit of a $k$-dimensional extension there corresponds $1/2^k n(n-1)\dots(n-k+1)$), after which these fractions are to be added.

4) The shares of the inner weight of the point $P$, found according to the rules indicated in items 1), 2), and 3) and corresponding to all its rays, must be summed. The number obtained as a result will be called the **inner weight of an integral point** $P$ as a point belonging to the skeleton of the polyhedron.

In the same way one can also compute the outer weight of the point $P$, by finding all extensions of various dimensions generated by this point and going strictly outside the polyhedron. It is not hard to see that the sum of the inner and outer weights of any integral vertex will be equal to its full weight, i.e., to one.

As for peripheral points of the skeleton that are not genuine vertices of the polyhedron, they can always be regarded as its non-genuine vertices and their inner weight can be computed, in essence, in exactly the same way as for genuine ones. Let us note that, in this case, for an integral point strictly belonging to an $(n-1)$-dimensional face of the polyhedron, its inner weight (as also its outer weight) will always be equal to $1/2$. For integral points belonging to faces lower than the $(n-1)$-st dimension, there are no such constant values of the inner weight of the point that would depend only on the dimen-

of a face cannot be indicated, since these values also depend on the position of the face.

The sum of the internal weights of all (both internal and peripheral) points of the frame of a polyhedron shall be called the **weight of the polyhedron**. It is not hard to conclude that the concept of the weight of a polyhedron, defined in this way, is additive in the sense that, if one "algebraically adds" two polyhedra (i.e., either unites two polyhedra having no common internal points into one, or subtracts from one polyhedron another that has no points external with respect to the first), then the new polyhedron will have weight equal to the corresponding algebraic sum of the weights of the two given ones.

Let us call, for an $n$-dimensional lattice, any point of space whose coordinates are only integers and halves of odd integers a **centering point**. Each such point will be a center of symmetry for the whole lattice. There are $n+1$ categories of centering points for each method of numbering the coordinate axes, beginning with points all of whose coordinates are integers and ending with points all of whose coordinates are halves of odd integers. We shall call an $n$-dimensional polyhedron **special** if it is positioned so that all its (true) vertices are centering points of one and the same category (i.e., the coordinates of all its true vertices with the same names are either simultaneously integers or simultaneously halves of odd integers).

**Theorem 1.** *Every $n$-dimensional special parallelepiped has weight equal to its $n$-dimensional volume.*

Let the volume of the given parallelepiped be $V$, and its weight $Q$, and let $Q = V + \alpha$. We must prove that $\alpha = 0$. Denote by $d$ the least integer not smaller than the length of the maximum diameter of the parallelepiped. Construct an $n$-dimensional cube with one-dimensional edges parallel to the coordinate axes and with vertices lying at those centering points all of whose coordinates are halves of odd integers. Denote the edge length of the cube by $l$. Then the volume of the cube will be $l^n$, and, as is not difficult to calculate, its weight will also be $l^n$. Construct this cube so that the given parallelepiped lies inside the cube. Fill this cube with parallelepipeds (obtained from the given one by parallel translations and adjacent to one another along whole $(n-1)$-dimensional faces) in such a way that every internal point of the cube is either an internal point of one and only one of the parallelepipeds, or a common peripheral point of adjacent parallelepipeds, and so that there are no superfluous parallelepipeds not needed for the filling. Construct also a second cube, concentrically homothetic to the first and having edge length $l + 2d$. All parallelepipeds of the filling of the first cube will lie strictly inside the second. Let the number of parallelepipeds in the filling be $p$. For the volumes we obtain the inequalities

$$l^n \leq pV < (l + 2d)^n.$$

Now let us note the essential circumstance that the weights of all the parallelepipeds in the filling of the cube are equal to one another. Indeed, a parallel translation of any polyhedron in which some vertex of it is carried from a centering point to a centering point of the same category cannot change the frame of the polyhedron and, consequently, cannot change its weight. This is precisely the case here, because the parallelepiped is given special. Therefore, by the additivity of the weight, we also obtain for the weights the inequalities

$$l^n \leq pQ < (l + 2d)^n.$$

From the inequalities for the volumes and weights we conclude that

$$p|Q - V| < 2^n d^n l^{n-1};$$

but $p \geq l^n/V$, and therefore

$$|\alpha| < q,$$

where $q$ is a constant independent of $l$. Hence $\alpha = 0$ (since the length $l$ can be taken arbitrarily large), and the theorem is proved. In substance, the theorem has been proved not only for parallelepipeds as convex polyhedra satisfying the known requirements for filling space, but also for nonconvex polyhedra satisfying the same requirements (since the convexity of the parallelepiped is not used in the proof).

Starting from Theorem 1, one can obtain a number of other results of both general and special character. I shall mention some of them.

**Theorem 2.** *If an $n$-dimensional special (convex) parallelohedron is cut by any $(n-1)$-dimensional plane passing through the center of symmetry of the parallelohedron into two polyhedra, then the weight of each of them is equal to its volume.*

This follows from the fact that, for a special parallelohedron, the center of symmetry will always be a centering point (not necessarily of the same category as the vertices); therefore it will be simultaneously both the center of symmetry of the parallelohedron and the center of symmetry of the lattice. Consequently, the integral points of the frameworks of both polyhedra are put into a one-to-one correspondence (by symmetry), together with their internal weights; that is, the weights of the two polyhedra are equal to one another and are equal to half the weight of the parallelohedron itself. But the same is true of their volumes, so the theorem is valid.

Calling such polyhedra **special**, from the additivity of weight and volume we immediately conclude that the following theorem holds.

**Theorem 3.** *The weight of every $n$-dimensional polyhedron composed by "algebraic addition" from special $n$-dimensional polyhedra is equal to its volume.*

The simplest special polyhedra are special $n$-dimensional "half-parallelepipeds," i.e., polyhedra obtained by cutting a special $n$-dimensional parallelepiped by an $(n-1)$-dimensional plane passing through some $(n-2)$-dimensional face of the parallelepiped and its center. In the case of two-dimensional space, the simplest special polygons are all special triangles, so that an entirely arbitrary special polygon also has weight equal to its area. Hence, in particular, follows the theorem recently proved by Ehrhart ([1]) on the number of integral points in polygons with integral vertices. Ehrhart restricts himself to non-self-intersecting polygons, whereas, by extending the notion of weight introduced here to self-intersecting polygons (just as is done for area), one can apply the results obtained by us to such polygons as well.

In the case of three-dimensional space, the simplest special polyhedra are all special triangular prisms, so that already in three-dimensional space not every special polyhedron has weight equal to its volume. A sufficient condition for equality of the weight and volume of a polyhedron here is the possibility of obtaining the polyhedron by "algebraic summation" of special triangular prisms. At the same time, one can indicate a class (admittedly rather narrow) of special tetrahedra for which the weight is equal to the volume; in the general case, however, the weight of a special tetrahedron is not equal to its volume. I shall mention one more generalization of Theorem 1.

**Theorem 4.** *If, for an $n$-dimensional parallelohedron, one of the proper vertices is placed at an entirely arbitrary point of $n$-dimensional space, while the others have coordinates differing from the corresponding coordinates of this vertex by integers, then the weight of the parallelohedron is equal to its volume.*

Theorem 2 no longer extends to this more general case.

It is self-evident that the proposed method of counting integral points can be transformed into a corresponding method for counting the number of integral solutions of certain special systems of linear inequalities with an arbitrary number of unknowns.

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## CITED LITERATURE

E. Ehrhart. C. R., **241** No. 11, 686 (1955).

*Note: Figure translations are in progress. See original paper for figures.*