



Soviet-era science, translated into English

MATHEMATICS

1958

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195801.78859>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

B. Ya. FALEVICH

A NEW METHOD OF PROVING INCOMPLETENESS THEOREMS FOR SYSTEMS WITH CARNAP' S RULE AND ITS APPLICATION TO THE QUESTION OF THE RELATIONSHIP BETWEEN CLASSICAL AND CONSTRUCTIVE ANALYSIS

(Presented by Academician P. S. Aleksandrov, 3 I 1958)

Rosser ⁽¹⁾ transferred the incompleteness theorems proved by Gödel to systems with a restricted Carnap rule of rank $< \omega^2$. In § 1 of the present paper these results are extended to systems containing a restricted Carnap rule of very substantial transfinite ranks. In § 2 the possibility of constructing a constructive model of classical analysis is considered.

§ 1. By transfinite induction on ν we define S_ν —a system with a restricted Carnap rule of rank ν . S_0 is a system consisting of the narrow predicate calculus, considered over arithmetic, supplemented by quantifiers over predicates with the corresponding* axioms and rules of inference (but without predicates of predicates), and by the axiom of reducibility—for every formula in the system there exists a predicate equivalent to it. If the system S_ν has been defined, then a formula \mathfrak{A} is regarded as derivable in the system $S_{\nu+1}$ if: 1) the formula \mathfrak{A} is derivable in the system S_ν ; 2) the formula $\mathfrak{A}(a)$ contains a free variable a and the formulas $\mathfrak{A}(n)$ are derivable in S_ν for every natural n (Carnap' s rule); 3) the formula is derivable from the formulas indicated in points 1), 2) by a finite number of applications of the rules of inference of the system S_0 . If ν is of the second kind, then

$$S_\nu = \sum_{\beta < \nu} S_\beta.$$

The results of the paragraph may be formulated as follows. Let $x < y$ denote a predicate of the system S_ν that well-orders the natural series according to the transfinite type ν . Suppose that ν is a complete** transfinite number and that there exists a transfinite number τ , less than ν , such that in the system S_τ : 1) it is provable that the predicate $x < y$ well-orders the natural series;

2) for any natural numbers n and m either the formula $n < m$ or the formula $\overline{n < m}$ is derivable (the predicate $x < y$ is decidable); 3) it is provable that ν is a complete transfinite number. Denote by δ the nearest complete transfinite number greater than τ . Then, if the system S_α , $\delta \leq \alpha \leq \nu$, is consistent, there exists in it a formula such that neither it nor its negation is derivable, and a formula expressing the consistency of the system S_α can be constructed in it, but is in the system S_α an undervivable formula.

The simplest and most important case is that in which $x < y$ is a primitive-recursive predicate and $\tau = 0$. Then, proceeding only from the assumptions:

I. In S_0 it is derivable that the predicate $x < y$ well-orders the natural

* Analogous to the axioms and rules of inference for quantifiers in the narrow calculus.

** A transfinite number ν is called complete if, for any $\sigma < \nu$, the set of transfinite numbers β , $\sigma \leq \beta < \nu$, and the set of transfinite numbers less than ν , considered as well-ordered sets, are similar.

series, theorem (1) is proved for all systems S_α , $6 \leq \alpha \leq \nu$. Below, for this simpler case, a complete construction of the provability predicates of the systems S_α is given, and the most important points of the proof are indicated. All predicates and functions considered below are primitive recursive.

Denote by $x <_1 y$ a predicate that well-orders the natural-number series according to the transfinite type μ , for which condition (1) is fulfilled. We shall assume that $(x)[0 \leq_1 x]$. Put*:

$$\varphi(x) = \left[\frac{x+1}{2^{\exp_0(x+1)}} \right] - 1,$$

$$x <_2 y \equiv (Ez_1)_{z_1 \leq x} (Ez_2)_{z_2 \leq y} \left\{ 2z_1 = x \ \& \ 2z_2 = y \ \& \ \left[\frac{x}{2} \right] <_1 \left[\frac{y}{2} \right] \right\},$$

$$x <_3 y = \{ \varphi(x) <_2 \varphi(y) \vee [\varphi(x) = \varphi(y) \ \& \ \exp_0(x+1) < \exp_0(y+1)] \}.$$

The predicate $x <_3 y$ well-orders the natural-number series according to the transfinite type $\nu = \omega \times \mu$, and assertion (1) holds for it; everywhere below the initial system is taken to be S_ν and the predicate $x <_3 y$.

$$x <_4 y \equiv (Ez_1)_{z_1 \leq x} (Ez_2)_{z_2 \leq y} \left\{ 2z_1 = x \ \& \ 2z_2 = y \ \& \ \left[\frac{x}{2} \right] <_3 \left[\frac{y}{2} \right] \right\}.$$

We shall denote the even numbers, ordered by the predicate $x <_4 y$ according to the transfinite type ν , by the letters $\alpha, \beta, \gamma, \dots$, and use them as indices for

denoting systems, for example S_α, S_β . The function $\alpha \oplus 1 = 4\alpha/2 + 2$ gives the transfinite number $\alpha \oplus 1$, immediately following α in the ordering $\beta <_4 \gamma$. $f(\alpha) = \alpha \oplus 1$, if $\alpha <_4 \alpha_\omega$; $f(\alpha) = \alpha$, if $\alpha_\omega \leq_4 \alpha$, where α_ω is the even number playing the role of the transfinite number ω in the ordering $\beta <_4 \gamma$.**

$$x <_5 y = \{\varphi(x) <_4 \varphi(y) \vee [\varphi(x) = \varphi(y) \& \exp_0(x+1) < \exp_0(y+1)]\}.$$

The predicate $x <_5 y$ well-orders the natural-number series according to the transfinite type $\omega \times \nu$. The even numbers α, β, \dots play, in this ordering, the role of transfinite numbers of the second kind. The function $\psi(x) = \exp_0[\exp_0(x+1)]$ will be a function of large span on each set of natural numbers $M_\alpha = \{n\}$, $\alpha \leq_5 n <_5 \alpha \oplus 1$.

Let us proceed to the construction of the provability predicates of the systems S_α and of the system S_ν . Suppose that an arithmetization has been constructed for the system S_0 . We take as known the following predicates and functions. $A(x)$ is the predicate: the number x is the number of an axiom of the system S_0 ; $F(x, y, z)$ is the predicate: the formula with number z directly follows, by the inference rules of the system S_0 , from the formulas with numbers x and y ; $s(x, j)$ is the function giving the number of the formula obtained from the formula with number x by replacing the free variable a by the number j . Denote

$$\Delta_1(y) \equiv A[\psi(y)];$$

$$\Delta_2(y_1, y_2, y, T) \equiv [y_1 <_5 y \& y_2 <_5 y \& \varphi(y_1) = \varphi(y_2) = \varphi(y) \& T(y_1) \& T(y_2) \&$$

$$\& F(\psi(y_1), \psi(y_2), \psi(y))];$$

$$\Delta_3(y', y, j, T) \equiv [\varphi(y') <_5 \varphi(y) \& T(y') \& \psi(y'') = s(\psi(y), j)];$$

$$U(x, T) \equiv (y) [y \leq_5 x \rightarrow \{T(y) \sim [\Delta_1(y) \vee (Ey_1)(Ey_2)\Delta_2(y_1, y_2, y, T)] \vee$$

$$\vee (j)(Ey'')\Delta_3(y'', y, j, T)\}];$$

$$B(x) \equiv (ET) U(x, T) \& T(x); \quad B(x, z) \equiv \psi(x) = z \& B(x);$$

$B(x, z)$ is the provability predicate of the system S_ν : the number x is a proof in S_ν of the formula with number z .

* For the functions x/y , $\exp_n(x)$, and $\text{long}(y)$, see (4).

** For odd numbers x one may take $f(x) = 0$.

We define, by transfinite induction on α , the provability predicates of the systems S_α , $B_\alpha(x, z)$. Put:

$$D_0(y) \equiv (i)_{i \leq \text{long}(y)} \{ [\Delta_1(\exp_i(y)) \vee (En)_{n < i} (Em)_{m < i} F(\psi[\exp_n(y)], \psi[\exp_m(y)], \vee[\exp_i(y)])] \&$$

$$\& \exp_i(y) <_5 f(0) \} \& (n)_{n \leq \text{long}(y)} (m)_{m \leq \text{long}(y)} [n < m \rightarrow \exp_n(y) <_5 \exp_m(y)];$$

$$B_0(x) \equiv (Ey)_{y \leq \zeta(x)} [D_0(y) \& \exp_{\text{long}(y)}(y) = x],$$

where $\zeta(x)$ is a certain definite function bounding the quantifier;

$$B_0(x, z) \equiv \psi(x) = z \& B_0(x).$$

If the predicates $D_\alpha(y)$, $B_\alpha(x)$, and $B_\alpha(x, z)$ have been constructed, then

$$D_{\alpha \oplus 1}(y) \equiv$$

$$\equiv (i)_{i \leq \text{long}(y)} \{ [\Delta_1(\exp_i(y)) \vee (En)_{n < i} (Em)_{m < i} F(\psi[\exp_n(y)], \psi[\exp_m(y)], \psi[\exp_i(y)])] \vee$$

$$\vee (j)(Ey') \Delta_3(y', \exp_i(y), j, B_\alpha) \} \& f(\alpha) \leq_5 \exp_i(y) <_5 f(\alpha \oplus 1) \} \&$$

$$\& (n)_{n \leq \text{long}(y)} (m)_{m \leq \text{long}(y)} [n < m \rightarrow \exp_n(y) <_5 \exp_m(y)];$$

$$B_{\alpha \oplus 1}(x) \equiv B_\alpha(x) \vee (Ey)_{y \leq \zeta(x)} [D_{\alpha \oplus 1}(y) \& \exp_{\text{long}(y)}(y) = x];$$

$$B_{\alpha \oplus 1}(x, z) \equiv \psi(x) = z \& B_{\alpha \oplus 1}(x).$$

If α is of the second kind, then

$$B_\alpha(x, z) \equiv \psi(x) = z \ \& \ B(x) \ \& \ x <_5 f(\alpha).$$

The proof of the theorem is based on the following lemmas:

Lemma 1. In S'_0 the following formulas are derivable:

$$1) \quad U(x, T_1) \ \& \ U(x, T_2) \rightarrow (y)\{y <_5 x \rightarrow [T_1(y) \sim T_2(y)]\};$$

$$2) \quad U(x, B);$$

$$3) \quad x <_5 f(\alpha) \rightarrow U(x, B_\alpha);$$

$$4) \quad B_\alpha(x) \sim B(x) \ \& \ x <_5 f(\alpha).$$

Lemma 2. The formula $B_\alpha(x, z)$ ($B(x, z)$) is decidable in the system S_α (S_ν).

Lemma 3. If from the formula with number n in S_α (in S_ν) without application of Carnap's rule a formula with number m is derivable, then in S_0 the formula

$$(Ex)B_\alpha(x, n) \rightarrow (Ex)B_\alpha(x, m) \quad ((Ex)B(x, n) \rightarrow (Ex)B(x, m))$$

is derivable.

Lemma 4. If the number r is the number of a formula; $e(z)$ is a function giving the number of the formula that is the negation of the formula with number z , then in $S_{0\oplus 1}$ the formulas

$$(Ex)B_\alpha(x, e(r)) \rightarrow (Ex)B_\alpha(x, e[s(r, y)]),$$

$$(Ex)B(x, e(r)) \rightarrow (Ex)B(x, e[s(r, y)])$$

are derivable.

Lemma 5. In the system $S_{0\oplus 6}$ (putting $0 \oplus (k + 1) = (0 \oplus k) \oplus 1$) for every number q the formulas

$$B_\alpha(x, q) \rightarrow (Eu)B_\alpha(u, s(r_\alpha, x)), \quad \text{where } r_\alpha \text{ is the number of the formula } B_\alpha(a, q);$$

$$B(x, q) \rightarrow (Eu)B(u, s(r, x)), \quad \text{where } r \text{ is the number of the formula } B(a, q).$$

are derivable.

Lemmas 3, 4, and 5 are, respectively, the 1st, 2nd, and 3rd conditions of applicability of Gödel's second theorem, indicated by Hilbert and Bernays in ⁽²⁾. The principal method of proof of the lemmas is transfinite induction.

The proof of the incompleteness theorems themselves for the systems S_α , $0 \leq \alpha \leq \omega$, and the system S_ν , is carried out according to the schemes given by Hilbert and Bernays in ⁽²⁾.

§ 2. In the present work, as a formalization of classical analysis, the system S_0 described above has been chosen. Such a choice is justified, since, on the one hand, the theory of the continuum can be constructed quite simply in this system by coding real numbers by predicates; on the other hand, set theory is considerably broader than the system S_0 . As an example of constructive analysis the system B of Ackermann ⁽³⁾ is taken. The principal result is the following theorem.

Theorem. In Ackermann's constructive analysis a proper model of the system S_0 cannot be constructed.

Below we formulate the notion of a proper model and indicate the method of proof of the theorem.

We shall call an axiomatic system \tilde{S} a **subsystem of the axiomatic system** A if the sets of terms and formulas of the system \tilde{S} are, respectively, subsets of the sets of terms and formulas of the system A . We shall call a subsystem \tilde{S} of the axiomatic system A a **model of the axiomatic system** S in the system A , if: 1) between the sets of terms and formulas of the system \tilde{S} and, respectively, the sets of terms and formulas of the system S , 2) between the axioms and rules of inference of the systems \tilde{S} and S , one can establish a one-to-one correspondence which is an isomorphism with respect to the rules of inference of the systems \tilde{S} and S .

Suppose that in the system S there have been constructed: 1) an arithmetization of the system S and the provability predicate of this system $B(x, z)$; 2) an arithmetization of the system A and the provability predicate of the system A , $B_A(x, z)$; 3) a function $e_A(n_A)$, which, from the number n_A of a formula of the system A , gives the number of the formula that is (in the sense of the system A) its negation.

A model \tilde{S} of the system S in the system A is called **proper** if the following four conditions are fulfilled: 1) in the system S one can construct the provability predicate of the system \tilde{S} , $\tilde{B}(x, z)$, and, for every natural number n , derive the formula $(\exists x)\tilde{B}(x, n) \rightarrow (\exists x)B_A(x, n)$; 2) if x is the number of a formula of the system S , and \tilde{x} is the number of the formula corresponding to this formula in the model \tilde{S} , then in the system S there exists a function $f(x)$ such that $f(x) = \tilde{x}$; 3) in the system S there exists a formula with number r , and in the system A there exists a derivable formula with number t_A , such that in S the

formulas $(Ex)B_A(x, t_A)$ and $f(r) = e_A(t_A)$ are provable; 4) if in the system S , for the number n , the formula $(Ex)\tilde{B}(x, f(n))$ is derivable, then the formula $(Ex)B(x, n)$ is also derivable.

The proof of the theorem relies only on conditions 1)–4) of the propriety of the model. First of all, in the system S_0 the provability predicate of Ackermann's system is constructed. Since Ackermann's constructive analysis is a system with a restricted Carnap rule of a certain transfinite rank, this can be done by the same method by which provability predicates were constructed above for the systems S_ν . It is then shown that Ackermann's proof of consistency, given for his system of constructive analysis, can be formally carried out in the system S_0 , i.e. that in S_0 a formula expressing the consistency of Ackermann's system is derivable.

The final stage of the proof is carried out by contradiction. Suppose that there exists a proper model \tilde{S}_0 of the system S_0 in Ackermann's constructive analysis. Then, by the properties of a proper model, from the formula expressing the consistency of Ackermann's system in S_0 , a formula is derivable expressing the consistency of the proper model \tilde{S}_0 , and from this latter formula a formula is derivable expressing the consistency of the system S_0 itself; and we arrive at a contradiction with the second incompleteness theorem for the system S_0 , which asserts that the formula expressing the consistency of the system S_0 is not derivable within this system.

Rybinsk Evening
Institute of Aviation Technology

Received
25 XII 1957

REFERENCES

1. B. Rosser, *J. Symb. Log.*, **2**, No. 3, 129 (1937).
2. D. Hilbert, P. Bernays, *Grundlagen der Mathematik*, Berlin, **1**, 1934; **2**, 1939.
3. W. Ackermann, *Math. Zs.*, **55**, No. 3, 364 (1952); **57**, No. 2, 155 (1953).
4. R. Peter, *Recursive Functions*, Moscow, 1954.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.