SovietRxiv

Soviet-era science, translated into English

MATHEMATICS

N. A. KARPOVA

1958

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-195801.78666

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

MATHEMATICS

N. A. KARPOVA

ON CONTACT CIRCUITS FOR MONOTONE FUNCTIONS

(Presented by Academician M. V. Keldysh on 25 VI 1958)

One of the basic problems of circuit theory is the synthesis of circuits that realize certain functions and contain the smallest possible number of elements. Depending on the class of circuit elements, circuits realizing one and the same function may be more or less complex in their topological structure, contain a greater or smaller number of elements, etc. It is known, for example, that a monotone function of the algebra of logic can be realized by a circuit containing only closing contacts. However, by bringing into consideration opening contacts, valves, memory elements, etc., one can obtain circuits which, in a certain sense, are simpler than the former.

Fig. 1

In the present note the complexity of realizing monotone functions in the class of contact circuits made of closing contacts is compared with their realization by circuits in which opening contacts are allowed, and it is shown that the class of circuits with arbitrary contacts is considerably simpler than the class of circuits consisting only of closing contacts.

Let us introduce an order relation between tuples of zeros and ones. Namely, we shall write
$(\alpha_1, \dots, \alpha_n) < (\beta_1, \dots, \beta_n)$ if, for all i, $1 \leq i \leq n$, the relations $\alpha_i \leq \beta_i$ are satisfied.

Definition. A function of the algebra of logic $f(x_1, \dots, x_n)$ is called **monotone** if for any tuples $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n)$ such that
$(\alpha_1, \dots, \alpha_n) < (\beta_1, \dots, \beta_n)$, one has
$f(\alpha_1, \dots, \alpha_n) \leq f(\beta_1, \dots, \beta_n)$.

It is clear that the realization of monotone functions in the class of circuits made of closing contacts is not the best from the point of view of the number of elements in the circuit. An example is known, constructed by Yu. L. Vasil' ev,

SovietRxiv

Fig. 2

Figure 2: Fig. 2

which shows that the minimal circuit made of closing contacts for the function

$$f = x_1(x_2 \vee x_3) \vee x_4(x_5 \vee x_6) \vee (x_2 \vee x_3)(x_5 \vee x_6)$$

is not minimal in the class of all contact circuits (Fig. 1a and 1b).

Let us consider the realization of monotone functions in the class of all contact circuits (class A) and in the class of circuits made of closing contacts (class B).

For comparing the complexity of circuits from the indicated classes it is convenient to introduce the following functions characterizing complexity: $L^+(f)$ – the number of contacts in a minimal circuit of class B realizing the given function f; $L(f)$ –the number of contacts in a minimal circuit of class A realizing the given function f.

To characterize the relative complexity of the classes under consideration, one also introduces

$$\lambda(f) = \frac{L^+(f)}{L(f)} \quad \text{and} \quad \lambda(n) = \max_{f \text{ of } n \text{ arguments}} \lambda(f).$$

Below a sequence of functions f_n (f_n depends on n arguments) will be constructed for which $\lambda(f_n)$ substantially exceeds one in the limit as $n \to \infty$.

Lemma 1. *In realizing the monotone function $f(a, b, a_1, b_1) = aa_1 \vee bb_1 \vee a_1b_1$ by a circuit from the class B, a minimal circuit contains 5 contacts; moreover, one of the contacts a_1 or b_1 must be repeated twice in the circuit.*

Proof. Suppose the contrary, i.e., that the relays a_1 and b_1 each contain one contact, and show that then the circuit must necessarily contain the chain ab.

Fig. 2

To realize the given function, the circuit must contain the chain a_1b_1; by assumption, it has length 2 (Fig. 2a). In addition, aa_1 and bb_1 must be realized. Since the contacts a_1 and b_1 are already contained in the circuit, aa_1 can be realized in a unique way (see Fig. 2b). For the same reason, bb_1 can be realized in a unique way (Fig. 2c). Consider the point of intersection of the chains a_1a and b_1b closest to the minus pole and not coinciding with the pole (denote this point by 2). There is at least one point of intersection of these chains–this is point 1. Consider the chain l_1l_2, where l_1 is the part of the chain aa_1 from the minus pole to point 2, and l_2 is the part of the chain bb_1 from the plus pole to point 2. Then l_1 can contain only contacts a, since the chain l_1 has length 1 and the contact a_1 is adjacent to the plus pole. The chain l_2 can contain only contacts b, since the single contact b_1 is adjacent to the minus pole and cannot enter l_2. Thus, the chain l_1l_2 realizes the conjunction ab, which is not contained

SovietRxiv

Fig. 3

Figure 3: Fig. 3

in the function. It remains to suppose that at least one of the contacts a_1 or b_1 enters the circuit more than once, and the circuit contains at least 5 contacts. A circuit with 5 contacts can be constructed as shown in Fig. 3, which completely proves Lemma 1.

Fig. 3

Lemma 2. *Let $\mathfrak{A} = x_1 \vee x_2 \vee \dots \vee x_n$, $\mathfrak{B} = y_1 \vee \dots \vee y_n$; then the monotone function*

$$f_{2n+2}(a, b, x_1, \dots, x_n, y_1, \dots, y_n) = a\mathfrak{A} \vee b\mathfrak{B} \vee \mathfrak{A}\mathfrak{B}$$

can be realized by a circuit from the class B containing no fewer than $3n + 2$ contacts.

Proof. Lemma 1 proved above allows us to assert that Lemma 2 is true for the case $n = 1$. Suppose that the assertion of Lemma 2 is valid for all $n < n_0$, and that for n_0 there is a lower estimate smaller than $3n_0 + 2$. Then it is obviously impossible to establish such a correspondence that, for every x_i from $\mathfrak{A}$, a relay containing one contact has a y_j from $\mathfrak{B}$ corresponding to it such that the corresponding relay contains at least two contacts, and conversely; i.e., under any correspondence there will be such x_i from $\mathfrak{A}$ and y_j from $\mathfrak{B}$ that the indicated relays in the circuit contain one contact each. But then, substituting the constant 0 in place of all x_l from $\mathfrak{A}$ and y_r from $\mathfrak{B}$ different from x_i and y_j, we obtain the function $f(a, b, x_i, y_j)$, which is realized by a circuit in which the distribution of contacts is different from that asserted in Lemma 1, namely, the relays x_i and y_j contain only one contact each.

But Lemma 1 is applicable to the function $f(a, b, x_i, y_i)$ obtained, and consequently we obtain a contradiction to the given assumption; the assertion of Lemma 2 is true for all n, and a minimal realization of the given function is a circuit of the form indicated in Fig. 4a or in Fig. 4b.

Lemma 3. Let $\mathfrak{A} = x_1 \vee \dots \vee x_n$, $\mathfrak{B} = y_1 \vee \dots \vee y_{n-1}$; then the monotone function

$$f_{2n+1}(a, b, x_1, \dots, x_n, y_1, \dots, y_{n-1}) = a\mathfrak{A} \vee a\mathfrak{B} \vee \mathfrak{A}\mathfrak{B}$$

can be realized by a circuit from class **B** containing no fewer than $3n$ contacts.

Proof. For the proof we shall use the result of Lemma 2. Suppose that the circuit contains fewer than $3n$ contacts. Then, substituting the constant 0 for x_n, we obtain a circuit realizing a function

Fig. 4

SovietRxiv

Fig. 4

Figure 4: Fig. 4

of the form indicated in Lemma 2, but for it the estimate $3n-1$ holds. Taking into account that the function under consideration depends essentially on x_n, we obtain that the circuit for the given function contains no fewer than $3n$ contacts. On the other hand, it can be realized with this number of contacts (Fig. 4b and 4a).

Theorem. There exists a sequence of functions f_n (f_n depends on n arguments) for which $\lambda(f_n) \geq 3/2$.

Proof. A sequence satisfying the condition of the theorem is given by the functions of the form f_{2n+1} and f_{2n+2} considered in Lemmas 2 and 3. As established above, for the realization of the indicated functions in class **B** there is a lower bound of $3n$ contacts.

In class **A** one can construct a circuit for the same function with no more than $2n+3$ contacts (Fig. 4c). Thus, in the class of circuits **A** there is an upper bound of $2n+3$ for the number of contacts. Directly from this, and also using the definition of $\lambda(f_n)$, for the given sequence of functions we obtain $\lambda(f_n) \geq 3/2$, as was required to prove.

Corollary. $\lambda(n) \geq 3/2$.

Received
24 VI 1958

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.