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Abstract

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MATHEMATICS

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ON THE DEPENDENCE OF THE INDEX OF AN OPERATOR OF A BOUNDARY-VALUE PROBLEM FOR AN ELLIPTIC SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER ON THE LEADING COEFFICIENTS

(Presented by Academician I. N. Vekua on 15 IV 1958)

This article is a natural continuation of work (1); here the notation and concepts introduced in (1) will be used. We consider the operator $\left(A\left(x, \frac{\partial}{\partial x}\right), B\left(y, \frac{\partial}{\partial x}\right)\right)$ of the type indicated in (1), and prove:

Theorem. The index of the operator $\left(A\left(x, \frac{\partial}{\partial x}\right), B\left(y, \frac{\partial}{\partial x}\right)\right)$ does not change under arbitrary changes of the leading coefficients of the operators $A\left(x, \frac{\partial}{\partial x}\right)$ and $B\left(y, \frac{\partial}{\partial x}\right)$, if in the course of these changes the following four conditions are not violated: the ellipticity condition [of the operator $A\left(x, \frac{\partial}{\partial x}\right)$; the condition

$$\det \sum_{i=1}^{n} B_i(y)\nu_i(y) \neq 0 \qquad (y \in S),$$

$\nu(y) = (\nu_1(y), \dots, \nu_n(y))$ is the unit vector of the inward normal to the surface S, bounding the domain V in which the boundary-value problem corresponding to the operator

$\left(A\left(x, \frac{\partial}{\partial x}\right), B\left(y, \frac{\partial}{\partial x}\right)\right)$
is considered; and the conditions of reducibility of the boundary-value problems corresponding to the operators
$\left(A\left(x, \frac{\partial}{\partial x}\right), B\left(y, \frac{\partial}{\partial x}\right)\right)$ and
$\left(A^*\left(x, \frac{\partial}{\partial x}\right), B^*\left(y, \frac{\partial}{\partial x}\right)\right)^*$
to regular integral equations (3).

Proof. It is based on the properties of the matrix $\Gamma(x, z)$, described in (1).

Let first the number l of linearly independent solutions of the boundary-value problem

$$A\left(x, \frac{\partial}{\partial x}\right) u(x) = 0 \qquad (x \in V); \tag{I_0}$$

$$\lim_{x \to y} B\left(y, \frac{\partial}{\partial x}\right) u(x) = 0 \qquad (y \in S) \tag{II_0}$$

be not less than the number m of linearly independent solutions of the problem

$$A^*\left(x, \frac{\partial}{\partial x}\right) v(x) = 0 \qquad (x \in V); \tag{I_0^*}$$

$$\lim_{x \to y} B^*\left(y, \frac{\partial}{\partial x}\right) v(x) = 0 \qquad (y \in S). \tag{II_0^*}$$

* The definition of the operator $\left(A^*\left(x, \frac{\partial}{\partial x}\right), B^*\left(y, \frac{\partial}{\partial x}\right)\right)$ see in (2).

Then the matrix

$$\Gamma_0(x, z) = \Gamma(x, z) + \sum_{k=1}^{m} u_k(x) v_k^*(z) \qquad (x, z \in V \cup S,\ x \neq z) \tag{1}$$

has all the properties of the matrix $\Gamma(x, z)$, except for the properties (2) in (^{1}), which are replaced by the following:

$$\begin{gathered} \int_S u_i^*(z) \Gamma_0^*(z, x)\, d_z S = \begin{cases} v_i(x) & (i = 1, \dots, m), \\ 0 & (i = m+1, \dots, l); \end{cases} \\ \int_S \Gamma_0(x, z) v_i(z)\, d_z S = u_i(x) \qquad (i = 1, \dots, m). \end{gathered} \tag{2}$$

Here, as in (^{1}), $u_1(x), \dots, u_l(x)$; $\mathfrak{u}_1(x), \dots, \mathfrak{u}_l(x)$ and $v_1(x), \dots, v_m(x)$, $\mathfrak{v}_1(x), \dots, \mathfrak{v}_m(x)$ are pairs of complete biorthogonal on S systems of linearly independent solutions of the problems (I_0), (II_0) and (I_0^*), (II_0^*), respectively. For non-operator matrices, by A^* one means the transposed matrix A.

Let $\hat{A}_{ij}(x)$ $(i, j = 1, \dots, n)$ be matrices of order p, twice continuously differentiable in $V \cup S$, and let $\hat{B}_i(y)$ $(i = 1, \dots, n)$ be matrices of order p, continuously differentiable along S, such that for the operator

$$\left(A\left(x, \frac{\partial}{\partial x}\right) + \sum_{i,j=1}^{n} \hat{A}_{ij}(x) \frac{\partial^2}{\partial x_i \, \partial x_j}, \quad B\left(y, \frac{\partial}{\partial x}\right) + \sum_{i=1}^{n} B_i(y) \frac{\partial}{\partial x_i} \right) \tag{3}$$

the first two conditions of the theorem are satisfied. Then, without loss of generality, one may assume that

$$\sum_{i,j=1}^{n} \hat{A}_{ij}(y) \nu_i(y) \nu_j(y) = \sum_{i=1}^{n} \hat{B}_i(y) \nu_i(y) = 0 \qquad (y \in S).$$

Therefore the solutions of the boundary-value problems

$$A\left(x, \frac{\partial}{\partial x}\right) \hat{u}(x) + \sum_{i,j=1}^{n} \hat{A}_{ij}(x) \frac{\partial^2 \hat{u}(x)}{\partial x_i \, \partial x_j} = 0 \qquad (x \in V); \tag{$\hat{I}_0$}$$

$$\lim_{x \to y} \left[B\left(y, \frac{\partial}{\partial x}\right) \hat{u}(x) + \sum_{i=1}^{n} \hat{B}_i(y) \frac{\partial \hat{u}(x)}{\partial x_i} \right] = 0 \qquad (y \in S); \tag{$\hat{I}I_0$}$$

$$A^*\left(x, \frac{\partial}{\partial x}\right) \hat{v}(x) + \sum_{i,j=1}^{n} \hat{A}_{ij}^*(x) \frac{\partial^2 \hat{v}(x)}{\partial x_i \, \partial x_j} + 2 \sum_{i,j=1}^{n} \frac{\partial \hat{A}_{ij}^*(x)}{\partial x_i} \frac{\partial \hat{v}(x)}{\partial x_j} + \sum_{i,j=1}^{n} \frac{\partial^2 \hat{A}_{ij}^*(x)}{\partial x_i \, \partial x_j} \hat{v}(x) = 0 \qquad (x \in V); \tag{$\hat{I}_0^*$}$$

$$\begin{aligned}\lim_{x \to y} &\left\{ B^*\left(y, \frac{\partial}{\partial x}\right) \hat{v}(x) + \sum_{i=1}^{n} \left[2 \sum_{k=1}^{n} \hat{A}_{ik}^*(y) \nu_k(y) - \hat{B}_i^*(y) \right] \frac{\partial \hat{v}(x)}{\partial x_i} \right\} \\ &+ \sum_{i,j=1}^{n} \left\{ \frac{\partial \hat{A}_{ij}^*(y)}{\partial y_i} + \frac{\partial \left[\left(\sum_{k=1}^{n} \hat{A}_{ik}^*(y) \nu_k(y) - \hat{B}_i^*(y) \right) \nu_j(y) \right]}{\partial y_i} \right. \\ &\qquad \left. - \frac{\partial \left[\left(\sum_{k=1}^{n} \hat{A}_{jk}^*(y) \nu_k(y) - \hat{B}_i^*(y) \right) \nu_i(y) \right]}{\partial y} \right\} \nu_j(y) \hat{v}(y) = 0 \qquad (y \in S)\end{aligned} \tag{$\hat{I}I_0^*$}$$

by the formulas

$$\hat{u}(x) = \int_V \Gamma_0(x,z)\, A\Big(z, \frac{\partial}{\partial z}\Big)\, \hat{u}(z)\, dz + \int_S \Gamma_0(x,z)\, B\Big(z, \frac{\partial}{\partial z}\Big)\, \hat{u}(z)\, d_{zS} + \sum_{k=1}^{l} u_k(x) \int_S u_k^*(z)\hat{u}(z)\, d_{zS},$$

$$\hat{v}(x) = \int_V \Gamma_0^*(z,x) F(z)\, dz + \int_S \Gamma_0^*(z,x) f(z)\, d_{zS} \qquad (x \in V \cup S)$$

are mapped from the solutions of the corresponding two pairs of singular integral equations; moreover, the first formula gives a one-to-one mapping, while by means of the second, in view of (2), exactly $l-m$ linearly independent solutions of the corresponding pair of singular equations pass into the trivial solutions of the problem (I_0^*), (II_0^*). For small matrices $\hat{A}_{ij}(x)$ $(i,j = 1,\dots,n;\ x \in V \cup S)$, $\hat{B}_i(y)$ $(i = 1,\dots,n;\ y \in S)$, the singular part of the kernel has a resolvent, and both pairs of singular integral equations are equivalently reduced to two adjoint pairs of regular equations.

Hence, under sufficient smoothness of the matrices $\hat{A}_{ij}(x)$ and $\hat{B}_i(y)$, and also under fulfillment, for the operator (3) and its adjoint, of the remaining two conditions of the theorem, its assertion is obtained easily. If $l < m$, instead of the matrix (1) one takes the matrix

$$\Gamma_0(x,z) = \Gamma(x,z) + \sum_{k=1}^{l} u_k(x) v_k^*(z) \qquad (x, z \in V \cup S;\ x \neq z).$$

Similarly one can show that the index of the operator

$$\left(A\Big(x, \frac{\partial}{\partial x}\Big),\, B\Big(y, \frac{\partial}{\partial x}\Big) \right)$$

does not change under a change of the surface S of the domain V, if in the course of this change the four conditions stated in the theorem are not violated.

Using this assertion, it is easy to verify that the index of the operator

$$\left(A\Big(\frac{\partial}{\partial x}\Big),\, B\Big(y, \frac{\partial}{\partial x}\Big) \right)$$

is equal to zero if the coefficients A_i $(i,j = 1,\dots,n)$ of the operator $A\big(\frac{\partial}{\partial x}\big)$ are constant, while the coefficients

$$B_i(y) = \sum_{j=1}^{n} B_{ij} \nu_j(y)$$

$$(i = 1, \dots, n), \qquad \det \sum_{i,j=1}^{n} B_{ij}\nu_i(y)\nu_j(y) \neq 0 \qquad (y \in S),$$

where all B_{ij} are constant (obviously, as in (1)), and here it is assumed that the boundary-value problems corresponding to the operators

$$\left(A\left(\frac{\partial}{\partial x}\right), B\left(y, \frac{\partial}{\partial x}\right)\right) \quad \text{and} \quad \left(A^*\left(\frac{\partial}{\partial x}\right), B^*\left(y, \frac{\partial}{\partial x}\right)\right)$$

are reducible to the regular integral equations of Ya. B. Lopatinskii).

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Note: Figure translations are in progress. See original paper for figures.

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