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Abstract

Full Text

Mathematics

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Theorems on the Mean Value and Multiplication of Summable Series

(Presented by Academician A. N. Kolmogorov on 13 II 1958)

1. Denote the set of convergent sequences* $x = \{x_\nu\}$ by c, and that of absolutely convergent sequences $\left(\sum |\bar{\Delta}_\nu x_\nu| < \infty\right)$ by a.

Let a series $\sum a_k$ be given and let

$$x = \{x_\nu\} = \left\{ \sum_{k=0}^{\nu} a_k \right\}.$$

We shall say that the series $\sum a_k$ is A-summable ($|A|$-summable) by the triangular method $A = (a_{n\nu})$ to the sum $A(x)$, if $\{A_n(x)\} \in c$ ($\{A_n(x)\} \in a$), where

$$A_n(x) = \sum_{\nu=0}^{n} a_{n\nu} x_\nu,$$

and if

$$\lim_{n\to\infty} A_n(x) = A(x).$$

The set of all A-summable ($|A|$-summable) series will be denoted by cA (aA). We shall say that the method A is regular if it sums all convergent series with preservation of sums, and that the method A is ac-regular (absolutely regular) if it sums ($|A|$-sums) all absolutely convergent series with preservation of sums.

Define, for the method A, the following estimates (where $x = \{x_\nu\}$ is an arbitrary sequence, $K = \text{const}$, and k' is some integer):

$$\left| \sum_{\nu=0}^{k} a_{n\nu} x_\nu \right| \le K \, |A_{k'}(x)| \qquad (0 \le k' \le k \le n); \tag{1}$$

$$\sum_{n=k+1}^{\infty} \left| \bar{\Delta}_n \sum_{\nu=0}^{k} a_{n\nu} x_\nu \right| \le K \, |A_{k'}(x)| \qquad (0 \le k' < k), \tag{2}$$

$$\sum_{\nu=0}^{n} |a_{n\nu} x_\nu| \le K \sum_{\nu=0}^{n} |\bar{\Delta}_\nu A_\nu(x)|. \tag{3}$$

Estimates (1) and (3) were first introduced by Jurkat and Peyerimhoff (1,2). They applied them to the study of the problem of summability factors and inclusion of summability methods. In the present note we formulate necessary and sufficient conditions, obtained with the aid of estimates (1)–(3), for the summability of the product of series.**

* If the limits of variation of the indices are not indicated, the indices take all integer values $0, 1, \ldots$. Below, everywhere,

$$\sum \quad \text{means} \quad \sum_{\nu=0}^{\infty}, \qquad \bar{\Delta}_n \alpha_n = \alpha_n - \alpha_{n-1} \quad (\alpha_{-1} = 0)$$

and

$$\Delta_n \alpha_n = \alpha_n - \alpha_{n+1}.$$

** Prof. Kangro pointed out to me the possibility of applying these estimates in the study of the summability of products of series.

Sufficient conditions for the fulfillment of (1) are given in (^{1}). If A satisfies estimate (1) and

$$\sum_{n=k+1}^{\infty} \sum_{\nu=0}^{k} \left| \bar{\Delta}_n \Delta_\nu \frac{a_n}{a_{k\nu}} \right| \leq M \left(a_{nk} \neq 0, \ \frac{a_{n,k+1}}{a_{k,k+1}} = 0 \text{ and } M = \text{const} \right),$$

then A satisfies estimate (2); if $\Delta_n |a_{n\nu}| = |\Delta_n a_{n\nu}|$, then A satisfies estimate (3). We shall say that A satisfies* MVT 1 (respectively MVT 2, MVT 3) if A satisfies estimate (1) (respectively (2), (3)).

2. Let the methods $A = (a_{n\nu})$ and $B = (b_{n\nu})$ be normal (i.e. triangular and $a_{nn} \neq 0, \ b_{nn} \neq 0$), let the method $C = (c_{n\nu})$ be triangular, and let

$$d_{n\nu} = \sum_{k=0}^{n-\nu} b_{n,k+\nu} a_{n,n-k}.$$

Consider the following conditions (in which all terms whose denominator is zero are excluded; $a_{nk} \neq 0, \ b_{nk} \neq 0$):

$$\sum_{k=0}^{n} \sum_{\mu=0}^{n-k} \sum_{\nu=0}^{k} |\Delta_\mu \Delta_\nu D_{nk\mu\nu}| \leq M_1, \tag{4}$$

where

$$D_{nk\mu\nu} = \frac{\Delta_\nu c_{n,\mu+\nu}}{d_{n,\mu+\nu}} \frac{a_{n,n-k+\nu}}{a_{k\nu}} \frac{b_{n,\mu+k}}{b_{n-k,\mu}};$$

$$\sum_{\nu=0}^{n-k}\left|\Delta_\nu \frac{\Delta_\nu c_{n,\nu+k}}{a_{nk} b_{n-k,\nu}}\right| \leq M_2; \tag{5}$$

$$\left|\frac{\Delta_\nu c_{n,\nu+k}}{a_{nk} b_{n-k,\nu}}\right| \leq M_3; \tag{6}$$

$$\sum_{\nu=0}^{n-k}\left|\Delta_\nu \frac{b_{n,\nu+k}}{b_{n-k,\nu}}\right| \leq M_4; \tag{7}$$

$$\sum_{n=k}^{\infty}\sum_{\nu=0}^{n-k}\left|\bar{\Delta}_n \Delta_\nu \frac{b_{n,\nu+k}}{b_{n-k,\nu}}\right| \leq M_5; \tag{8}$$

$$\left|\frac{b_{nn}}{b_{kk}}\right| \leq M_6 \quad (k \leq n); \tag{9}$$

$$\sum_{n=k}^{\infty}\left|\bar{\Delta}_n \frac{b_{nn}}{b_{n-k,n-k}}\right| \leq M_7, \tag{10}$$

the method is regular and satisfies MVT 1;

(11)

the method is *ac*-regular and satisfies MVT 1;

(12)

the method is absolutely regular and satisfies MVT 2;

(13)

the method is absolutely regular and satisfies MVT 2 and 3**

(14)

3. Let the series $\sum u_k$ and $\sum v_k$ be given. From these series form the product series by Cauchy’ s rule, i.e. the series $\sum w_k$, where

$$w_k = \sum_{\nu=0}^{k} v_{k-\nu} u_\nu.$$

The following theorems 1-4 can be proved (where the methods A and B are normal, and C is triangular).

* MVT —mean value theorem.
** Conditions (11), (12), (13), or (14) will in what follows be imposed on the methods A and B.

Theorem 1. Let A and B satisfy condition (11), and C—condition (4). The series $\sum w_k$, for arbitrary series $\sum u_k \in cA$ and $\sum v_k \in cB$, is C-summable to the sum $C(W) = A(U)B(V)$ if and only if C is regular.

Theorem 2. Let A satisfy condition (14), B—condition (11), and C—condition (5). The series $\sum w_k$, for arbitrary $\sum u_k \in aA$ and $\sum v_k \in cB$, is C-summable to the sum $C(W) = A(U)B(V)$ if and only if C is regular.

Theorem 3. Let A and B satisfy condition (14), and C—condition (6). The series $\sum w_k$, for arbitrary $\sum u_k \in aA$ and $\sum v_k \in aB$, is C-summable to the sum $C(W) = A(U)B(V)$ if and only if C is ac-regular.

Theorem 4. Let B satisfy conditions (7) and (12), ((8) and (13)). The series $\sum w_k$, for all $\sum u_k \in a$, is B-summable ($|B|$-summable) to the sum

$$B(W) = B(V) \sum u_k$$

if and only if $\sum v_k$ is B-summable ($|B|$-summable).

The proof of these theorems is based on the application of properties possessed by summability methods satisfying TC3, which allow one, from the boundedness of the sequence $\{C_n(W)\}$, to conclude its convergence. A similar method, as applied to the study of inclusion of summability methods, is described in detail in (1). The proof of Theorem 4 can be reduced to the application of necessary and sufficient conditions for ac-regularity and absolute regularity. For example, in applying estimate (2), the following property of the method A is used: if A is absolutely regular and satisfies TC3 2, then A is absolutely perfect, i.e. such that for any $\varepsilon > 0$ and $x \in aA$ with $A(x) = 0$ there exists a $g = \{g_\nu\}$ ($g_\nu = 0$, if $\nu > \nu_0$), such that

$$\|x - g\| = \sum |\Delta_n A_n(x - g)| < \varepsilon.$$

4. For certain summability methods in Theorems 1 and 4 the requirement of satisfying TC3 turns out to be superfluous. These include methods A, B, and C satisfying the conditions

$$\Delta_\nu \frac{\Delta_\nu c_{n\nu}}{d_{n\nu}} = 0 \quad (\nu < n), \tag{15}$$

$$\Delta_\nu \frac{a_{n,\nu+k}}{a_{n-k,\nu}} = 0, \qquad \Delta_\nu \frac{b_{n,\nu+k}}{b_{n-k,\nu}} = 0 \quad (\nu < n-k). \tag{16}$$

Then, for example, instead of Theorem 1 the following is proved:

Theorem 5. Let A, B, and C satisfy conditions (15) and (16). The series $\sum w_k$, for arbitrary $\sum u_k \in cA$, $\sum v_k \in cB$, is C-summable to the sum $C(W) = A(U)B(V)$ if and only if the methods $T = (t_{nk})$ and $T' = (t_{n,n-k})$ are regular, where $t_{nk} = c_{nn}/a_{kk}b_{n-k,n-k}$.

Remark. If the method B is ac-regular (absolutely regular) and satisfies the second condition (16), then in Theorem 4 the conditions (7) and (11) may be replaced by (9) ((8) and (13) may be replaced by (10)).

5. For the application of Theorems 1-4 in the case of Riesz methods, we indicate that the Riesz method (R, a_ν) (i.e. the method $A = (a_{n\nu})$, where $a_{n\nu} = a_\nu/A_n$, if $\nu \leq n$

$$\left(A_n = \sum_{\nu=0}^{n} a_\nu \neq 0, \quad a_\nu \neq 0 \right),$$

and $a_{n\nu} = 0$, if $\nu > n$) satisfies: a) estimate (1) if and only if $|A_k| \leq K|A_n|$ $(k \leq n)$; b) estimate (2)

then and only then, when $\sum\limits_{n=k+1}^{\infty} \left| \frac{a_n A_k}{A_n A_{n-1}} \right| \ll K$; c) the estimate (3) then and only then, when $\sum\limits_{\nu=0}^{n} |a_\nu| \ll K|A_n|$.

These conditions are satisfied if the method (R, a_n) is: a) ac-regular; b) absolutely regular; c) regular. Thus, in the case of the Riesz method, under conditions (11)–(14), the requirement that the method satisfy TC3 1 or TC3 2 proves to be superfluous, and instead of TC3 3 we may require that the method be regular. It is now not difficult to write down Theorems 1–4 for the case in which A and B are Riesz methods.

The Voronoi–Nörlund method (WN, a_ν) (i.e. the method $A \equiv (a_{n\nu})$, where $a_{n\nu} = a_{n-\nu}/A_n$, if $\nu \leq n$, and $a_{n\nu} = 0$, if $\nu > n$) satisfies the estimates (1)–(3) only under very narrow conditions. However, (WN, a_ν) always satisfies the first condition (16). If another method (WN, b_ν) is also given, then the method (WN, c_ν), where

$$c_\nu = \sum_{k=0}^{\nu} a_{\nu-k} B_k,$$

satisfies condition (15), and from Theorem 5 we obtain one well-known theorem of Mears (^{3}, Theorem 2). From Theorem 4, taking the remark into account, we can also obtain Theorems 10′ and 11′ of Mears from (^{4}).

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References

[1] W. Jurkat, A. Peyerimhoff, *Math. Zs.*, **55**, 92 (1951).
[2] A. Peyerimhoff, *Math. Zs.*, **57**, 265 (1953).
[3] F. M. Mears, *Bull. Am. Math. Soc.*, **41**, 875 (1935).
[4] F. M. Mears, *Ann. Math.*, (2), **44**, 401 (1943).

Note: Figure translations are in progress. See original paper for figures.