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Abstract

Full Text

MATHEMATICS

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ON THE GENERALIZATION OF A THEOREM OF MOORE

(Presented by Academician S. L. Sobolev, 25 IV 1958)

A sequence $\{\varepsilon_k\}$ is called a sequence of convergence multipliers of the second (first) kind with respect to a given set of series $\mathfrak{X}$, if every series* $\sum x_k$ from $\mathfrak{X}$ generates a convergent series $\sum \varepsilon_k x_k$ ($\sum \varepsilon_k(x_0 + x_1 + \cdots + x_k)$). In the case of absolute convergence of the series $\sum \varepsilon_k x_k$ ($\sum \varepsilon_k(x_0 + x_1 + \cdots + x_k)$), one speaks of multipliers of absolute convergence of the second (first) kind.

Let $A = (a_{nk})$ be a normal matrix, i.e. $a_{nk} = 0$ for $k > n$, $a_{kk} \neq 0$. A series $\sum x_k$ is said to be A-summable (A-bounded, $|A|$-summable) if the sequence $\{x'_n\}$, defined by the formula

$$x'_n = \sum_{k=0}^{n} a_{nk} x_k, \qquad (n = 0, 1, \ldots), \tag{1}$$

converges (is bounded, is absolutely convergent).

In the case of the Voronoi–Nörlund summability method, for which $a_{nk} = P_n^{-1} P_{n-k}$ ($\{P_k\}$ is a given sequence, $P_k \neq 0$), Moore ([1]) proved that if the Voronoi–Nörlund method is regular and satisfies the condition

$$\sum k\,|d_k| < \infty, \tag{2}$$

where d_k is determined from the relation

$$\sum k\,|d_k| < \infty,$$

then $\{\varepsilon_k\}$ is a sequence of convergence multipliers of the second kind with respect to the set of series summable by the Voronoi–Nörlund method if and only if:

1) $\varepsilon_k = O(P_k^{-1})$, $(k = 0, 1, \ldots)$,

2) $\sum \left| P_k \sum_{\nu=k}^{\infty} d_{\nu-k} \varepsilon_\nu \right| < \infty.$

In the present note Moore' s theorem is generalized to the case when x_k are elements of some Banach space X; ε_k are bounded linear operators from X into a Banach space Y, and (a_{nk}) is any normal numerical matrix subject to the condition

$$\sum^{\nu} D_\nu < \infty, \tag{3}$$

* Unless otherwise stated, the summation indices run through all nonnegative integer values $0, 1, \ldots$.

where $D_\nu = \sup\limits_k |a_{\nu+k,\nu+k} a'_{\nu+k,k}|$, $(a'_{nk}) = (a_{nk})^{-1}$. In this case the set of all A-summable * (A-bounded, $|A|$-summable) series in X is denoted simply by A ($A_0, |A|$).

Theorem 1. *In order that $\{\varepsilon_k\}$ be a sequence of convergence multipliers of the second kind with respect to A, it is necessary and sufficient that the following conditions be fulfilled:*

1) $\sum a'_k \varepsilon_k x$ converges for $x \in X$;
2) $\|\varepsilon_k\| = O(a_{kk})$ $(k = 0, 1, \ldots)$;
3) $\left\| \sum\limits_{k=0}^{p} A_k x_k \right\| = O(1)$ $(\|x_k\| \leqslant 1;\ p = 0, 1, \ldots)$,

where

$$a'_k = \sum_{\nu=0}^{k} a'_{k\nu}, \qquad A_k = \sum_{\nu=k}^{\infty} a'_{\nu k} \varepsilon_\nu.$$

Proof. Since we have

$$\sum_{k=0}^{n} \varepsilon_k x_k = \sum_{k=0}^{n} A_{nk} x'_k,$$

where x'_k is defined by formula (1) and

$$A_{nk} = \sum_{\nu=k}^{n} a'_{\nu k} \varepsilon_\nu,$$

it follows that $\{\varepsilon_k\}$ is a sequence of convergence multipliers with respect to A if and only if the transformation

$$y_n = \sum_{k=0}^{n} A_{nk} x'_k \qquad (n = 0, 1, \ldots) \tag{4}$$

takes every convergent sequence $\{x'_k\} \subset X$ into a convergent sequence $\{y_n\} \subset Y$. For this, according to ([2]), it is necessary and sufficient that:

a) there exist $\lim_{n\to\infty} A_{nk}x = A_{kx}$ $(x \in X;\ k = 0, 1, ...)$;

b) there exist $\lim_{n\to\infty} \sum_{k=0}^{n} A_{nk}x$ $(x \in X)$;

c)

$$\left\|\sum_{k=0}^{p} A_{nk}x_k\right\| = O(1)$$

$$(\|x_k\| \leqslant 1;\ p \leqslant n;\ n = 0, 1, ...).$$

Condition b) is equivalent to condition 1 of Theorem 1. From c) it follows that $\|A_{nn}\| = O(1)$, i.e. condition 2). Taking a) into account, from c), as $n \to \infty$, we obtain condition 3) of Theorem 1.

Conversely, from condition 2) on the basis of (3) follows the fulfillment of condition a). Further, in view of condition 2, we find

$$\sum_{k=0}^{p} \|A_k - A_{pk}\| = \sum_{k=0}^{p} \left\| \sum_{\nu=p+1-k}^{\infty} a'_{\nu+k,k}\varepsilon_{\nu+k} \right\| \leqslant$$

$$\leqslant M \left(\sum_{\nu=1}^{p+1} \sum_{k=p+1-\nu}^{p} D_\nu + \sum_{\nu=p+2}^{\infty} \sum_{k=0}^{p} D_\nu \right) < M \sum \nu D_\nu,$$

where M is a constant. By virtue of (3), now from condition 3) follows c). Theorem 1 is proved.

Let us note that if $a_{n0} = 1$ $(n = 0, 1, ...)$, then $a'_k = 0$ for $k > 0$, $a'_0 = 1$, and condition 1) of Theorem 1 is fulfilled. In particular, if A is a regular

* Limit processes in X (and also in Y) are understood in the sense of strong convergence.

the Voronoi-Nörlund summation method, then (2) ensures the fulfillment of condition (3), and from Theorem 1 one obtains a generalization of Moore' s theorem to abstract convergence sets.

Theorem 1 is also applicable if A is the summation method of weighted arithmetic means ([3]). Then condition (3) is fulfilled if A is regular and satisfies the condition $P_{k+1}^{-1}p_{k+1} = O(P_k^{-1}p_k)$.

Using the theorems giving necessary and sufficient conditions for (4) to transform every bounded, respectively absolutely convergent, sequence of the space X into a convergent sequence of the space Y, or every absolutely convergent sequence of the space X into an absolutely convergent sequence of Y ([4]), one can prove the validity of the following two theorems. As before, it is assumed that the matrix $A = (a_{nk})$ is normal and satisfies condition (3).

Theorem 2. In order that $\{\varepsilon_k\}$ be a sequence of convergence factors of the second kind with respect to A_0, it is necessary and sufficient that the following conditions be satisfied:

1) $\|\varepsilon_k\| = o(a_{kk})$ $(k = 0, 1, ...)$;
2) $\sum A_k x_k$ converges uniformly for $\|x_k\| \leq 1$.

Theorem 3. In order that $\{\varepsilon_k\}$ be a sequence of factors of (absolute) convergence of the second kind with respect to $|A|$, it is necessary and sufficient that the following conditions be satisfied:

1) $\sum a'_k \varepsilon_k x$ converges (absolutely) for $x \in X$;
2) $\|\varepsilon_k\| = O(a_{kk})$ $(k = 0, 1, ...)$.

Remark. Theorems 1-3 turn out to be valid for factors of convergence, respectively of absolute convergence of the first kind, if in these theorems (and also in condition (3)) a_{nk} is replaced by $a_{nk} - a_{n,k+1}$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.